Virasoro constraints for Kontsevich-Hurwitz partition function

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Abstract: In [1, 2] M.Kazarian and S.Lando found a 1-parametric interpolation between Kontsevich and Hurwitz partition functions, which entirely lies within the space of KP τ-functions. In [3] V.Bouchard and M.Marino suggested that this interpolation satisfies some deformed Virasoro constraints. However, they described the constraints in a somewhat sophisticated form of AMM-Eynard equations [4–7] for the rather involved Lambert spectral curve. Here we present the relevant family of Virasoro constraints *explicitly*. They differ from the conventional continuous Virasoro constraints in the simplest possible way: by a constant shift $\frac{u^2}{24}$ of the $\hat{L}_{-1}$ operator, where u is an interpolation parameter between Kontsevich and Hurwitz models. This trivial modification of the string equation gives rise to the entire deformation which is a conjugation of the Virasoro constraints $\hat{L}_m \to \hat{U}\hat{L}_m\hat{U}^{-1}$ and "twists" the partition function, $\mathcal{Z}_{\text{KH}} = \hat{U}Z_K$. The conjugation $\hat{U} = \exp\left\{\frac{u^2}{3}(\hat{N}_1 - \hat{L}_1) + O(u^6)\right\} = \exp\left\{\frac{u^2}{12}\left(\sum_k T_k\partial/\partial T_{k+1} - \frac{g^2}{2}\partial^2/\partial T_0^2\right) + O(u^6)\right\}$ is expressed through the previously unnoticed operators like $\hat{N}_1 = \sum_k(k+1)^2 T_k\partial/\partial T_{k+1}$ which annihilate the quasiclassical (planar) free energy $F_K^{(0)}$ of the Kontsevich model, but do not belong to the symmetry group $GL(\infty)$ of the universal Grassmannian.

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Contents

1. Introduction

Modern quantum field theory, nicknamed string theory [8], looks for a unified approach to seemingly different problems in different branches of sciences. In some areas it is already quite successful, for example, in the field of enumerative geometry. String theory usually formulates combinatorial problems in terms of generating functions and then interpret them as partition functions, i.e. as elements of certain D-modules, satisfying some sets of differential equations and usually possessing various ("dual") integral representations, often matrix or even functional. Therefore, these partition functions acquire a "hidden symmetry", with respect to change of integration variables, which manifests itself through rich integrability properties, which are already revealed in many examples. These examples begin from matrix models [9], in particular, from the celebrated Kontsevich model [10, 11], and from that on spread in many different directions. Hidden integrability is now found, as was originally predicted, in a vast variety of problems, both in physics and mathematics, and today it is accepted as an important and universal phenomenon. However, the underlying D-module structure, i.e. the set of constraints imposed on partition functions, is often ignored and not enough effort is given to identify and investigate it in each concrete example, what obscures the common Lie algebra origin of all these seemingly different situations.

In this paper, we take as an example the currently popular deformation of the Kontsevich model, used to describe the Hurwitz numbers (characterizing combinatorics of certain ramified coverings of a Riemann sphere) and Hodge integrals over the moduli space of complex curves. This Kontsevich-Hurwitz model is very interesting and a number of spectacular results is already obtained about it. In particular, the KP integrability of the model is already established. We are not going to enter any details about the model but one: our goal is to describe an underlying deformation of the "continuous Virasoro constraints" [12],[1] [13], which control the original Kontsevich model and should undoubtedly control its Kontsevich-Hurwitz generalization. In [3][2] it was actually suggested that the constraints remain Virasoro and bilinear, however, a somewhat sophisticated machinery of the AMM-Eynard equations [4 – 7] was used to discuss the issue. In our opinion, this formalism is very useful for a variety of purposes, both conceptual and technical (see [4, 5] for explanation of our views on this issue), but the Ward identities in a given model should be better formulated in a more direct and straightforward form, and we provide some evidence that such a form is indeed available: see eq. (2.9) below.

The paper is organized as follows. We start in section 2 with formulating the main statements discussed in the paper, in particular, introduce a set of deformed Virasoro constraints which are satisfied by the Kontsevich-Hurwitz partition function. In order to check that these Virasoro constraints, indeed, have something to do with the Kontsevich-Hurwitz partition function, in section 3 we discuss the general structure of solution to the the constraints, find its first terms solving the Virasoro constraints iteratively, and compare the results with the expansion of the Kontsevich-Hurwitz partition function obtained

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[1]See also [9] for many details about continuous Virasoro constraints.
[2]This paper is based on a set of ideas developed in [7] and [39].

in [1][3]–[3]. In section 4, known properties of the (generating functions of the) Hurwitz numbers and Hodge integrals, and interrelations between them are discussed, mainly basing on [1, 2]. We specially concentrate on three different sets of time variables ($\{p_k\}$, $\{T_k\}$ and $\{q_k\}$) in order to reveal better integrable properties behind these quantities, which are of our special interest in this section. In particular, it is realized that the integrability found in [2] in terms of q-variables has an easy and immediate explanation in terms of equivalent hierarchies. At last, in section 5 we demonstrate that the deformed Virasoro algebra proposed in the paper follows from the AMM-Eynard equation of [3]. The appendix is devoted to a diagram technique for calculating the Hurwitz numbers implied by formulae of section 4.

2. The main statement: eqs.(2.9)-(2.14)

According to [2], the Kontsevich-Hodge free energy is the double expansion

$$\mathcal{F}(T) = \sum_{q=0}^{\infty} u^{2q} F_q(T) = \sum_{p=0} g^{2p} F^{(p)}(T), \qquad (2.1)$$

$$F_q(T) = \sum_{p \geq q}^{\infty} g^{2p} F_q^{(p)}(T),$$

$$F^{(p)}(T) = \sum_{q=0} u^{2q} F_q^{(p)}(T)$$

where each

$$F_q^{(p)}(T) = (-)^q \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1,\ldots,k_n=0}^{\infty} \delta\left(\sum_{i=1}^{n}(k_i - 1) - (3p - 3 - q)\right) I_q^{(p)}(k_1,\ldots,k_n) T_{k_1} \ldots T_{k_n} \qquad (2.2)$$

is a generating function for the Hodge integrals [14, 15]

$$I_q^{(p)}(k_1,\ldots,k_n) = \int_{\mathcal{M}_{p,n}} \lambda_q \psi_1^{k_1} \ldots \psi_n^{k_n}. \qquad (2.3)$$

This definition, together with the ELSV formula [16], allowed M.Kazarian in [2] to relate $\mathcal{F}(T)$ to the generating function $H(p)$ of the Hurwitz numbers [17], which has a simple alternative representation [18] in terms of discrete Virasoro and W-operators [20, 21]

$$e^{H(p)} = e^{u^3 \hat{W}_0} e^{p_1}, \qquad (2.4)$$

where

$$\hat{W}_0 = \frac{1}{2} \sum_{m=1}^{\infty} p_m \hat{V}_m = \frac{1}{2} \sum_{i,j \geq 1} \left((i+j)p_i p_j \frac{\partial}{\partial p_{i+j}} + ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j}\right) \qquad (2.5)$$

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[3]For original proof of Witten's conjecture [34] with the help of Kontsevich's reformulation [10] see [13] and [11].

and $\hat{V}_m$ are the discrete Virasoro operators ($p_k = kt_k$)

$$\hat{V}_m = \sum_{k=1}^{\infty}(k+m)p_k\frac{\partial}{\partial p_{k+m}} + \sum_{i+j=m} ij\frac{\partial^2}{\partial p_i \partial p_j}\,. \tag{2.6}$$

This allows one to call $\mathcal{F}(T)$ the Kontsevich-Hodge-Hurwitz or simply Kontsevich-Hurwitz free energy. We return to Kazarian's construction in section 4.3, and now switch to an alternative description.

Its starting point is the fact that $F_0(T)$, the ordinary Kontsevich free energy, satisfies the "continuous Virasoro" constraints

$$\begin{aligned}
\hat{L}_m Z_0 &= 0 \quad \text{for } m \geq -1\,,\\
\hat{L}_m &= -\frac{\partial}{\partial \tau_{m+1}} + \sum_{k=\delta_{m,-1}}^{\infty}\left(k+\frac{1}{2}\right)\tau_k\frac{\partial}{\partial \tau_{k+m}}\\
&\quad + \frac{g^2}{8}\sum_{k=0}^{m-1}\frac{\partial^2}{\partial \tau_k \partial \tau_{m-1-k}} + \frac{\tau_0^2}{2g^2}\delta_{m,-1} + \frac{1}{16}\delta_{m,0}\,,\\
Z_0 &= \exp\left(\frac{1}{g^2}F_0(T)\right)\,,\\
T_k &= \frac{(2k+1)!!}{2^k}\tau_k = \frac{\Gamma(k+\frac{3}{2})}{\Gamma(\frac{3}{2})}\tau_k\,.
\end{aligned} \tag{2.7}$$

It is well known that, while the Virasoro constraints look more elegant in terms of τ-variables, their solutions such as Z and F, are seriously simplified if expressed through T-variables. Sometimes one denotes $\tau_k = \check{\tau}_{2k+1}$ to emphasize that $\{\tau_k\}$ are only half of all the time-variables in the Generalized Kontsevich Model (GKM) [11]: this is also reflected in the fact that Z_0 is a KdV τ-function [11], while the generic Z_{GKM} belongs to the KP family. According to [2], this is also true for the Kontsevich-Hurwitz partition function we consider in this paper, $\mathcal{Z} = \exp(\frac{1}{g^2}\mathcal{F})$: in appropriate variables (called $\check{q}_k$ in section 4.3 below) $\mathcal{Z}$ is a KP τ-function, and reduces to the KdV one only for $u = 0$. As usual, we often denote the first two terms (the linear-in-derivatives piece) in the Virasoro operator $\hat{L}_m$ through $\hat{l}_m$. The deformation of the Kontsevich model is described by a parameter u so that the Kontsevich model corresponds to $u = 0$, while the Hurwitz partition function [17], analyzed in some detail in [3], corresponds to $u = 1$.

Our claim, parallel to a rather implicit suggestion of [3], is that the full Kontsevich-Hurwitz partition function,

$$\mathcal{Z} = \prod_{q=0}^{\infty} Z_q = \exp\left(\frac{1}{g^2}\sum_q^{\infty}F_q\right) = \exp\left(\frac{1}{g^2}\sum_{p\geq q\geq 0}^{\infty}u^{2q}g^{2p}F_q^{(p)}(T)\right) = \exp\left(\frac{1}{g^2}\mathcal{F}(T)\right) \tag{2.8}$$

satisfies a set of deformed continuous Virasoro constraints. We claim that the relevant deformation is actually a *conjugation* [5]

$$\begin{aligned}
\hat{\mathcal{L}}_m \mathcal{Z} &= 0 \quad \text{for } m \geq -1\,,\\
\hat{\mathcal{L}}_m &= \hat{U}\hat{L}_m\hat{U}^{-1}
\end{aligned} \tag{2.9}$$

which obviously preserves structure of the Virasoro (sub)algebra. It follows that

$$\mathcal{Z} = \hat{U} Z_0 \tag{2.10}$$

is obtained by a simple twisting of the Kontsevich partition function (note also that, according to [2], $\mathcal{Z}$ in proper variables is a KP τ-function at any given value of u, for Z_0 this was originally proved in [11] and for $\mathcal{Z}$ this follows from (2.4) and the theory of equivalent hierarchies [22–24]). The operator $\hat{U}$ is explicitly given by[4]

$$\hat{U} = \exp\left\{ -\frac{u^2}{3}\left(\hat{L}_1 - \hat{N}_1\right) + O(u^6)\right\} = \exp\left\{\frac{u^2}{12}\left(\sum_k \tilde{T}_k \frac{\partial}{\partial T_{k+1}} - \frac{g^2}{2}\frac{\partial^2}{\partial T_0^2}\right) + O(u^6)\right\} \tag{2.12}$$

where $\tilde{T}_k = T_k - \delta_{k,1}$ are "shifted times" and

$$\hat{N}_1 = \sum_{k=0}^{\infty} (k+1)^2 \tilde{T}_k \frac{\partial}{\partial T_{k+1}} \tag{2.13}$$

is a new (for Kontsevich-model theory) operator, which annihilates the genus-zero free energy,

$$\hat{N}_1 F_0^{(0)} = 0 \tag{2.14}$$

and gives rise to an infinite family of such annihilators. Commutation relations between $\hat{L}_m$ and $\hat{N}_1$ imply that the lowest deformed Virasoro constraints act on $\mathcal{Z}$ as

$$\hat{\mathcal{L}}_{-1} = \hat{L}_{-1} - \frac{u^2}{24} \tag{2.15}$$

and

$$\hat{\mathcal{L}}_0 = \hat{L}_0 + u^2 \frac{\partial}{\partial(u^2)} \tag{2.16}$$

see the detailed explanation in sections 5.4–5.7, especially formula (5.41) and the discussions at the very end of section 5.7.

In what follows we recursively define coefficients in the free energy expansion from our suggested Virasoro constraints: see eqs. (3.12)–(3.16) below. In particular, we reproduce in this way all the terms, explicitly calculated in [3]. Note that non-trivial is already the property that conjugation operator (2.12) generates only contributions with $p \geq q$ to the logarithm of $\mathcal{Z}$.

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[4]The full operator $\hat{U}$ with higher order terms included is, in fact, of the form

$$\hat{U} = \exp\left(\sum_k s_k u^{4k+2} \hat{M}_{2k+1}\right), \quad \hat{M}_{2k+1} \equiv \sum_l \tilde{T}_l \frac{\partial}{\partial T_{l+k}} - \frac{1}{2}\sum_{a+b=2k}(-)^a \frac{\partial^2}{\partial T_a \partial T_b} \tag{2.11}$$

where the coefficients $s_k = \frac{B_{k+1}}{k(k+1)}$ are expressed through the Bernoulli numbers, $\sum_{k=1}^{\infty} \frac{B_{2k}x^{2k}}{(2k)!} \equiv \frac{x}{1-e^{-x}} - 1 - \frac{x}{2}$. This formula is found in [25, 26]. We are indebted to M.Kazarian for pointing out reference [25] which, in its turn, is based on refs. [27, 28] and for providing us with the text of his unpublished paper [26].

After that we perform the transformation [2] from T to p variables,

$$T_k = u^{2k+1} \sum_{n=1}^{\infty} \frac{n^{n+k}}{n!} u^{3n} p_n \tag{2.17}$$

and demonstrate that the answer coincides with $H(p)$ in (2.4) modulo necessary subtractions of certain p-linear and p-quadratic terms (the rooted and double-rooted tree contributions to $H(p)$):

$$\begin{aligned}
\mathcal{F}(T(p)) &= H(p) - H_{01}(p) - H_{02}(p) \\
&= H(p) - \sum_{n \geq 1} \frac{n^{n-2}}{n!} p_n u^{3(n-1)} - \frac{1}{2} \sum_{m,n \geq 1} \frac{m^m n^n}{(m+n) m! n!} p_m p_n u^{3(m+n)} \,.
\end{aligned} \tag{2.18}$$

Note that (2.4) and (2.17) involve *odd* powers of u, i.e. semi-integer powers of the deformation parameter u^2, but all odd powers drop away from the answer. Finally, we explain the relation between the AMM-Eynard equations of [3] and our conjugation of the continuous Virasoro algebra.

3. Solving Virasoro constraints

This is a standard procedure, so we do not go into too many details.

First, the deformed $\hat{\mathcal{L}}_0$ constraint (2.16) implies that the T_1 dependence can be defined exactly: from

$$(\hat{l}_0 + q) F_q = -\frac{1}{16} \delta_{q,0} \tag{3.1}$$

it follows that

$$F_q^{(p)} = \sum_{m=0}^{\infty} \sum_{k_1,\ldots,k_m \geq 0, \neq 1}^{\infty} \delta \left(\sum_{i=1}^{m} (k_i - 1)(3p - 3 - q) \right) J_q^{(p)}[k_1, \ldots, k_m] \frac{T_{k_1} \ldots T_{k_m}}{(1 - T_1)^{2p-2+m}} \,. \tag{3.2}$$

The r.h.s. in (3.1) is taken into account by a peculiar contribution to $F_0^{(1)}$, which deviates from (3.2): see the first term in eq. (3.13) below.

Second, the string equation (i.e. the $\hat{\mathcal{L}}_{-1}$ constraint) is satisfied separately by every constituent free energy $F_q^{(p)}$:

$$\hat{l}_{-1} F_q^{(p)} = \left((T_1 - 1) \frac{\partial}{\partial T_0} + \sum_{k=1}^{\infty} T_{k+1} \frac{\partial}{\partial T_k} \right) F_q^{(p)} = \xi_q^{(p)}(T) \tag{3.3}$$

where the only non-zero ξ's are

$$\xi_0^{(0)} = -\frac{1}{2} T_0^2 \quad \text{and} \quad \xi_1^{(1)} = \frac{1}{24} = -\gamma_1^{(1)} \,. \tag{3.4}$$

Moreover, together with p, q-dependent selection rules in (3.2) this equation defines the series $F_q^{(p)}(T)$ up to a finite number of free coefficients (we call them γ-parameters), one per each T_0-independent monomial allowed by selection rules.

The role of all remaining Virasoro constraints is just to fix these remaining undefined coefficients in front of different solutions of (3.3).

3.1 Solving string equation

3.1.1 $F_0^{(0)}$ — the genus-zero component of Kontsevich free energy

Since $\beta_0^{(0)} = -\frac{1}{2}T_0^2$, we have, after explicit resolution of the selection rules,

$$
\begin{aligned}
F_0^{(0)} &= \frac{1}{6} \cdot \frac{T_0^3}{1-T_1} + \sum_{s=1}^{\infty} \left(\sum_{k_1,\ldots,k_s \geq 2}^{\infty} J_0^{(0)}[k_1,\ldots,k_s] \frac{T_0^{3+\sum_{i=1}^s (k_i-1)} T_{k_1} \ldots T_{k_s}}{(1-T_1)^{1+\sum_{i=1}^s k_i}} \right) = \\
&= \frac{1}{6} \cdot \frac{T_0^3}{1-T_1} + \sum_{k \geq 2} J[k] \frac{T_0^{k+2} T_k}{(1-T_1)^{k+1}} + \sum_{k_1,k_2 \geq 2} J[k_1,k_2] \frac{T_0^{k_1+k_2+1} T_{k_1} T_{k_2}}{(1-T_1)^{k_1+k_2+1}} \\
&\quad + \sum_{k_1,k_2,k_3 \geq 2} J[k_1,k_2,k_3] \frac{T_0^{k_1+k_2+k_3} T_{k_1} T_{k_2} T_{k_3}}{(1-T_1)^{k_1+k_2+k_3+1}} + \cdots
\end{aligned}
\tag{3.5}
$$

(for simplicity we omitted the labels $p,q = 0,0$ in the last two lines). Substitution into (3.3) gives

$$
\begin{aligned}
0 &= \frac{1}{6} \frac{T_0^3 T_2}{(1-T_1)^2} - \sum_{k \geq 2}(k+2)J[k] \frac{T_0^{k+1} T_k}{(1-T_1)^k} + \sum_{k \geq 2} J[k] \frac{T_0^{k+2} T_{k+1}}{(1-T_1)^{k+1}} \\
&\quad + \sum_{k \geq 2}(k+1)J[k] \frac{T_0^{k+2} T_2 T_k}{(1-T_1)^{k+2}} - \sum_{k_1,k_2 \geq 2}(k_1+k_2+1)J[k_1,k_2] \frac{T_0^{k_1+k_2} T_{k_1} T_{k_2}}{(1-T_1)^{k_1+k_2}} \\
&\quad + \sum_{k_1,k_2 \geq 2} J[k_1,k_2] \frac{T_0^{k_1+k_2+1}(T_{k_1+1} T_{k_2} + T_{k_1} T_{k_2+1})}{(1-T_1)^{k_1+k_2+1}} \\
&\quad + \sum_{k_1,k_2 \geq 2}(k_1+k_2+1)J[k_1,k_2] \frac{T_0^{k_1+k_2} T_2 T_{k_1} T_{k_2}}{(1-T_1)^{k_1+k_2+2}} \\
&\quad + \sum_{k_1,k_2,k_3 \geq 2} J[k_1,k_2,k_3] \left(-(k_1+k_2+k_3) \frac{T_0^{k_1+k_2+k_3-1} T_{k_1} T_{k_2} T_{k_3}}{(1-T_1)^{k_1+k_2+k_3}} \right. \\
&\quad \left. + \frac{T_0^{k_1+k_2+k_3}(T_{k_1+1} T_{k_2} T_{k_3} + T_{k_1} T_{k_2+1} T_{k_3} + T_{k_1} T_{k_2} T_{k_3+1})}{(1-T_1)^{k_1+k_2+k_3+1}} \right) + \cdots .
\end{aligned}
$$

The structure of recurrent relations is obvious from these formulas. The first line implies that $J[2] = \frac{1}{24}$ and

$$
J[k] = \frac{1}{k+2} J[k-1] = \frac{1}{(k+2)!} \quad \text{for} \quad k \geq 2 .
\tag{3.6}
$$

The second line implies that

$$
\begin{aligned}
J[2,2] &= \frac{3}{5}J[2] = \frac{1}{40}, \\
J[2,k] &= \frac{1}{2(k+3)}\Big((k+1)J[k] + 2J[2,k-1]\Big) = \frac{(k-1)(k+4)}{2(k+3)!} \quad \text{for} \quad k \geq 3, \\
J[k_1,k_2] &= \frac{1}{k_1+k_2+1}\Big(J[k_1-1,k_2] + J[k_1,k_2-1]\Big) \quad \text{for} \quad k_1,k_2 \geq 3
\end{aligned}
\tag{3.7}
$$

and so on.

3.1.2 Generic $F_q^{(p)}$

Now we are ready to proceed to the case of generic q and p. Generic version of (3.5) is

$$
F_q^{(p)} = \sum_{s=1}^{\infty} \left(\sum_{\substack{k_1,\ldots,k_s \geq 2 \\ \sum_{i=1}^{s}(k_i-1) \geq 3p-3-q}}^{\infty} J_q^{(p)}[k_1,\ldots,k_s] \frac{T_0^{q+3-3p+\sum_{i=1}^{s}(k_i-1)}T_{k_1}\ldots T_{k_s}}{(1-T_1)^{q+1-p+\sum_{i=1}^{s}k_i}} \right) =
$$

$$
= \sum_{k \geq 3p-2-q} J[k] \frac{T_0^{k+2+q-3p}T_k}{(1-T_1)^{k+1+q-p}} + \sum_{\substack{k_1,k_2 \geq 2 \\ k_1+k_2 \geq 3p-1-q}} J[k_1,k_2] \frac{T_0^{k_1+k_2+1+q-3p}T_{k_1}T_{k_2}}{(1-T_1)^{k_1+k_2+1+q-p}}
$$

$$
+ \sum_{\substack{k_1,k_2,k_3 \geq 2 \\ k_1+k_2+k_3 \geq 3p-q}} J[k_1,k_2,k_3] \frac{T_0^{k_1+k_2+k_3+q-3p}T_{k_1}T_{k_2}T_{k_3}}{(1-T_1)^{k_1+k_2+k_3+1+q-p}} + \cdots . \tag{3.8}
$$

Substitution into the string equation (3.3) gives

$$
0 = - \sum_{k \geq 3p-2-q} (k+2+q-3p)J[k] \frac{T_0^{k+1+q-3p}T_k}{(1-T_1)^{k+q-p}} + \sum_{k \geq 3p-2-q} J[k] \frac{T_0^{k+2+q-3p}T_{k+2+q-3p}}{(1-T_1)^{k+1+q-p}}
$$

$$
+ \sum_{k \geq 3p-2-q} (k+1+q-p)J[k] \frac{T_0^{k+2+q-3p}T_2 T_k}{(1-T_1)^{k+2+q-p}}
$$

$$
- \sum_{\substack{k_1,k_2 \geq 2 \\ k_1+k_2 \geq 3p-1-q}} (k_1+k_2+1+q-3p)J[k_1,k_2] \frac{T_0^{k_1+k_2+q-3p}T_{k_1}T_{k_2}}{(1-T_1)^{k_1+k_2+q-p}}
$$

$$
+ \sum_{\substack{k_1,k_2 \geq 2 \\ k_1+k_2 \geq 3p-1-q}} J[k_1,k_2] \frac{T_0^{k_1+k_2+1+q-3p}(T_{k_1+1}T_{k_2} + T_{k_1}T_{k_2+1})}{(1-T_1)^{k_1+k_2+1+q-p}}
$$

$$
+ \sum_{\substack{k_1,k_2 \geq 2 \\ k_1+k_2 \geq 3p-1-q}} (k_1+k_2+1+q-p)J[k_1,k_2] \frac{T_0^{k_1+k_2+1+q-3p}T_2 T_{k_1}T_{k_2}}{(1-T_1)^{k_1+k_2+2+q-p}}
$$

$$
+ \sum_{\substack{k_1,k_2,k_3 \geq 2 \\ k_1+k_2+k_3 \geq 3p-q}} J[k_1,k_2,k_3] \left(-(k_1+k_2+k_3+q-3p) \frac{T_0^{k_1+k_2+k_3-1+q-3p}T_{k_1}T_{k_2}T_{k_3}}{(1-T_1)^{k_1+k_2+k_3+q-p}} \right.
$$

$$
\left. + \frac{T_0^{k_1+k_2+k_3+q-3p}(T_{k_1+1}T_{k_2}T_{k_3} + T_{k_1}T_{k_2+1}T_{k_3} + T_{k_1}T_{k_2}T_{k_3+1})}{(1-T_1)^{k_1+k_2+k_3+1+q-p}} \right) + \cdots .
$$

This time the first line implies that

$$
J_q^{(p)}[k] = \frac{1}{k+2+q-3p} J_q^{(p)}[k-1] \tag{3.9}
$$

$$
= \frac{1}{(k+2+q-3p)!} J_q^{(p)}[3p-2-q] \quad \text{for} \quad k \geq \max(3p-2-q, 2) .
$$

Since $p \geq q$, one has $3p - 2 - q \geq 2q - 2$ and separate consideration is needed only for a few cases: $(q,p) = (0,0), (0,1)$ and $(1,1)$, these are the only cases when terms which depend only on T_0 and T_1 can arise.

The second line implies that

$$J_q^{(p)}[2,2] = \frac{3+q-p}{(5+q-3p)!}J_q^{(p)}[2] \quad \text{for} \quad 3p-5-q \le 0\,. \tag{3.10}$$

Such terms exist only in five cases: $(q,p) = (0,0),\ (0,1),\ (1,1),\ (1,2)$ and $(2,2)$, moreover, in the case $(1,2)$ the T_2^2 term is independent of T_0 and enters with an independent coefficient, which is not fixed by the string equation (only by the higher Virasoro constraints): there is no $J[2]$ in this case to constrain $J[2,2]$.

Except for a few exceptional cases, the coefficients $J[2,k]$ are expressed through two independent (at the level of the string equation) parameters:

$$\begin{aligned}
J_q^{(p)}[2,k] &= \frac{k+1+q-p}{2(k+3+q-3p)}J_q^{(p)}[k] + \frac{1}{k+3+q-3p}J_q^{(p)}[2,k-1] = \\
&= \frac{(k+1+q-p)+(k+q-p)+\cdots+(k+2+q-p-j)}{2(k+3+q-3p)!} \\
&\quad + \frac{(k+3+q-3p-j)!}{(k+3+q-3p)!}J[2,k-j] = \\
&= \frac{k+q+p}{4(k+2+q-2p)!}J_q^{(p)}[3p-2-q] + \frac{1}{(k+3+q-3p)!}J_q^{(p)}[2,3p-3-q]\,.
\end{aligned}$$

As usual,

$$\begin{aligned}
J_q^{(p)}[k_1,k_2] &= \frac{1}{k_1+k_2+1+q-3p}\Big(J_q^{(p)}[k_1-1,k_2] + J_q^{(p)}[k_1,k_2-1]\Big) \tag{3.11}\\
&\quad \text{for} \quad k_1,k_2 \ge \max(3p-3-q,3)
\end{aligned}$$

and so on.

3.2 The first terms of F expansion

Putting different pieces together, one obtains

$$
\begin{aligned}
F_0 = &\underbrace{\left\{\tfrac{1}{6}\cdot\tfrac{T_0^3}{1-T_1} + \sum_{k\ge2}\tfrac{1}{(k+2)!}\tfrac{T_0^{k+2}T_k}{(1-T_1)^{k+1}} + \tfrac{1}{40}\cdot\tfrac{T_0^5 T_2^2}{(1-T_1)^5} + \sum_{k\ge3}\tfrac{(k+1)(k+2)}{2(k+3)!}\cdot\tfrac{T_0^{k+3}T_2 T_k}{(1-T_1)^{k+3}} + \cdots\right\}}_{\text{genus }0} \\[4pt]
&+\tfrac{1}{24}\underbrace{\left\{\big(1-\log(1-T_1)\big) + \sum_{k\ge2}\tfrac{1}{(k-1)!}\tfrac{T_0^{k-1}T_k}{(1-T_1)^k} + \tfrac{T_0^2 T_2^2}{(1-T_1)^4} + \sum_{k\ge3}\tfrac{k^2+k+2}{2k!}\tfrac{T_0^k T_2 T_k}{(1-T_1)^{k+2}} + \cdots\right\}}_{\text{genus }1} \\[4pt]
&+\underbrace{\big(\gamma_{01}^{(2)}=\tfrac{1}{9\cdot128}\big)\left\{\sum_{k\ge4}\tfrac{1}{(k-4)!}\left(\tfrac{T_0^{k-4}T_k}{(1-T_1)^{k-1}} + \tfrac{k+2}{2}\tfrac{T_0^{k-3}T_2 T_k}{(1-T_1)^{k+1}}\right) + \cdots\right\}}_{\text{genus }2} \\[4pt]
&+\underbrace{\big(\gamma_{02}^{(2)}=\tfrac{29}{45\cdot128}\big)\left\{\sum_{k\ge3}\tfrac{1}{(k-3)!}\tfrac{T_0^{k-3}T_2 T_k}{(1-T_1)^{k+1}} + \cdots\right\} + \big(\gamma_{03}^{(2)}=\tfrac{7}{3\cdot128}\big)\left\{\tfrac{T_2^3}{(1-T_1)^5} + \cdots\right\}}_{\text{genus }2} \\[4pt]
&+\underbrace{\big(\gamma_{01}^{(3)}=\tfrac{1}{9^2\cdot1024}\big)\left\{\sum_{k\ge7}\tfrac{1}{(k-7)!}\left(\tfrac{T_0^{k-7}T_k}{(1-T_1)^{k-2}} + \tfrac{k+3}{2}\cdot\tfrac{T_0^{k-6}T_2 T_k}{(1-T_1)^k}\right) + \cdots\right\} + \gamma_{02}^{(3)}\left\{\sum_{k\ge6}\tfrac{1}{(k-6)!}\tfrac{T_0^{k-6}T_2 T_k}{(1-T_1)^k} + \cdots\right\}}_{\text{genus }3} \\[4pt]
&+\underbrace{\gamma_{01}^{(4)}\left\{\sum_{k\ge10}\tfrac{1}{(k-10)!}\left(\tfrac{T_0^{k-10}T_k}{(1-T_1)^{k-3}} + \tfrac{k+4}{2}\cdot\tfrac{T_0^{k-9}T_2 T_k}{(1-T_1)^{k-1}}\right) + \cdots\right\} + \gamma_{02}^{(4)}\left\{\sum_{k\ge9}\tfrac{1}{(k-9)!}\tfrac{T_0^{k-9}T_2 T_k}{(1-T_1)^{k-1}} + \cdots\right\}}_{\text{genus }4} \\[4pt]
&+\underbrace{\cdots}_{\text{higher genera}} \tag{3.12}
\end{aligned}
$$

$$F_1 = \left(\gamma_1^{(1)} = -\tfrac{1}{24}\right) \underbrace{\left\{\tfrac{T_0}{1-T_1} + \sum_{k\geq 2}\tfrac{1}{k!}\tfrac{T_0^k T_k}{(1-T_1)^{k+1}} + \tfrac{1}{2}\cdot\tfrac{T_0^3 T_2^2}{(1-T_1)^5} + \sum_{k\geq 3}\tfrac{k+2}{2k!}\tfrac{T_0^{k+1} T_2 T_k}{(1-T_1)^{k+3}} + \cdots\right\}}_{\text{genus 1}}$$

$$+ \underbrace{\left(\gamma_{11}^{(2)} = -\tfrac{1}{15\cdot 32}\right)\left\{\sum_{k\geq 3}\tfrac{1}{(k-3)!}\left(\tfrac{T_0^{k-3} T_k}{(1-T_1)^k} + \tfrac{k+3}{2}\cdot\tfrac{T_0^{k-2} T_2 T_k}{(1-T_1)^{k+2}}\right) + \cdots\right\} + \left(\gamma_{12}^{(2)} = -\tfrac{5}{9\cdot 128}\right)\left\{\sum_{k\geq 2}\tfrac{1}{(k-2)!}\tfrac{T_0^{k-2} T_2 T_k}{(1-T_1)^{k+2}} + \cdots\right\}}_{\text{genus 2}}$$

$$+ \underbrace{\left(\gamma_{11}^{(3)} = -\tfrac{7}{5\cdot 27\cdot 1024}\right)\left\{\sum_{k\geq 6}\tfrac{1}{(k-6)!}\left(\tfrac{T_0^{k-6} T_k}{(1-T_1)^{k-1}} + \tfrac{k+4}{2}\cdot\tfrac{T_0^{k-5} T_2 T_k}{(1-T_1)^{k+1}}\right) + \cdots\right\} + \gamma_{12}^{(3)}\left\{\sum_{k\geq 5}\tfrac{1}{(k-5)!}\tfrac{T_0^{k-5} T_2 T_k}{(1-T_1)^{k+1}} + \cdots\right\}}_{\text{genus 3}}$$

$$+ \underbrace{\gamma_{11}^{(4)}\left\{\sum_{k\geq 9}\tfrac{1}{(k-9)!}\left(\tfrac{T_0^{k-9} T_k}{(1-T_1)^{k-2}} + \tfrac{k+5}{2}\cdot\tfrac{T_0^{k-8} T_2 T_k}{(1-T_1)^k}\right) + \cdots\right\} + \gamma_{12}^{(4)}\left\{\sum_{k\geq 8}\tfrac{1}{(k-8)!}\tfrac{T_0^{k-8} T_2 T_k}{(1-T_1)^k} + \cdots\right\}}_{\text{genus 4}}$$

$$+ \underbrace{\cdots}_{\text{higher genera}} \tag{3.13}$$

$$F_2 = \left(\gamma_2^{(2)} = \tfrac{7}{45\cdot 128}\right)\underbrace{\left\{\sum_{k\geq 2}\tfrac{1}{(k-2)!}\tfrac{T_0^{k-2} T_k}{(1-T_1)^{k+1}} + 3\tfrac{T_0 T_2^2}{(1-T_1)^5} + \sum_{k\geq 3}\tfrac{(k+1)(k+2)}{2(k-1)!}\tfrac{T_0^{k-1} T_2 T_k}{(1-T_1)^{k+3}} + \cdots\right\}}_{\text{genus 2}}$$

$$+ \underbrace{\left(\gamma_{21}^{(3)} = \tfrac{41}{7\cdot 81\cdot 1024}\right)\left\{\sum_{k\geq 5}\tfrac{1}{(k-5)!}\left(\tfrac{T_0^{k-5} T_k}{(1-T_1)^k} + \tfrac{k+5}{2}\cdot\tfrac{T_0^{k-4} T_2 T_k}{(1-T_1)^{k+2}}\right) + \cdots\right\} + \gamma_{22}^{(3)}\left\{\sum_{k\geq 4}\tfrac{1}{(k-4)!}\tfrac{T_0^{k-4} T_2 T_k}{(1-T_1)^{k+2}} + \cdots\right\}}_{\text{genus 3}}$$

$$+ \underbrace{\gamma_{21}^{(4)}\left\{\sum_{k\geq 8}\tfrac{1}{(k-8)!}\left(\tfrac{T_0^{k-8} T_k}{(1-T_1)^{k-1}} + \tfrac{k+6}{2}\cdot\tfrac{T_0^{k-7} T_2 T_k}{(1-T_1)^{k+1}}\right) + \cdots\right\} + \gamma_{22}^{(4)}\left\{\sum_{k\geq 7}\tfrac{1}{(k-7)!}\tfrac{T_0^{k-7} T_2 T_k}{(1-T_1)^{k+1}} + \cdots\right\}}_{\text{genus 4}}$$

$$+ \underbrace{\cdots}_{\text{higher genera}} \tag{3.14}$$

$$F_3 = \underbrace{\left(\gamma_{31}^{(3)} = -\tfrac{31}{5\cdot 27\cdot 1024}\right)\left\{\sum_{k\geq 4}\tfrac{1}{(k-4)!}\left(\tfrac{T_0^{k-4} T_k}{(1-T_1)^{k+1}} + \tfrac{k+6}{2}\tfrac{T_0^{k-3} T_2 T_k}{(1-T_1)^{k+3}}\right) + \cdots\right\} + \gamma_{32}^{(3)}\left\{\sum_{k\geq 3}\tfrac{1}{(k-3)!}\tfrac{T_0^{k-3} T_2 T_k}{(1-T_1)^{k+3}} + \cdots\right\}}_{\text{genus 3}}$$

$$+ \underbrace{\gamma_{31}^{(4)}\left\{\sum_{k\geq 7}\tfrac{1}{(k-7)!}\left(\tfrac{T_0^{k-7} T_k}{(1-T_1)^k} + \tfrac{k+7}{2}\cdot\tfrac{T_0^{k-6} T_2 T_k}{(1-T_1)^{k+2}}\right) + \cdots\right\} + \gamma_{32}^{(3)}\left\{\sum_{k\geq 6}\tfrac{1}{(k-6)!}\tfrac{T_0^{k-6} T_2 T_k}{(1-T_1)^{k+2}} + \cdots\right\}}_{\text{genus 4}}$$

$$+ \underbrace{\cdots}_{\text{higher genera}} \tag{3.15}$$

$$F_4 = \underbrace{\gamma_{41}^{(4)}\left\{\sum_{k\geq 6}\tfrac{1}{(k-6)!}\left(\tfrac{T_0^{k-6} T_k}{(1-T_1)^{k+1}} + \tfrac{k+8}{2}\cdot\tfrac{T_0^{k-5} T_2 T_k}{(1-T_1)^{k+3}}\right) + \cdots\right\} + \gamma_{42}^{(3)}\left\{\sum_{k\geq 5}\tfrac{k+1}{(k-5)!}\tfrac{T_0^{k-5} T_2 T_k}{(1-T_1)^{k+3}} + \cdots\right\}}_{\text{genus 4}}$$

$$+ \underbrace{\cdots}_{\text{higher genera}}$$

$$+ \cdots \tag{3.16}$$

We omitted the factors g^{2p} in these formulas, they can be immediately restored.

These expressions satisfy the string equation (3.3) for arbitrary values of γ-parameters. Actual values of γ's are given in brackets in the above formulas. There are many more

γ-parameters than can be seen in these lines: they appear in front of T_0-independent terms which have powers in T higher than shown in these formulas. γ-parameters can be defined in different ways. In our approach, they are dictated by higher Virasoro constraints (2.9).

In practice, *we derived (3.13)–(3.16) with all the proper values of γ-parameters with the help of (2.10)*: by acting with the explicitly known operator (2.12) on the known expression (3.12) for the Kontsevich partition function Z_0 (see, for example, the second paper of [5, appendix A1.2]). Now we are going to demonstrate that the free energy (3.12)–(3.16) is indeed the same as that considered in [1, 2] and [3].

3.3 Consistency with [1, 2]

It is an easy MAPLE exercise to check that substitution of $T(p)$ from (2.17) into $\mathcal{F}(T)$ reproduces $H(p, u)$ in (2.4):

$$H(p, u) - H_{01}(p, u) - H_{02}(p, u) = \sum_{p \geq q \geq 0} u^{2q} F_q^{(p)}\Big(T(p) \Big). \tag{3.17}$$

Of course, this demonstration is not a conceptual proof, which should be based on relating the Virasoro constraints (2.9) to the ones imposed on $\exp\Big(H(p) \Big)$. Such a proof seems straightforward, but it is left beyond the scope of the present paper. For some more details see section 4 below.

One comment, is, however, necessary already at this point. The Hurwitz function (2.4) and, thus, relation (3.17) are so far defined for $g^2 = 1$. One can restore the g^2-dependence, making use of the homogeneity property

$$F_q^{(p)}(\lambda^{2k-2} T_k) = \lambda^{6p-6-2q} F_q^{(p)}(T_k). \tag{3.18}$$

It follows that

$$\sum_{p \geq q \geq 0} \lambda^{6(p-1)} u^{2q} F_q^{(p)}(T_k) = \sum_{p \geq q \geq 0} (\lambda u)^{2q} F_q^{(p)}(\lambda^{2k-2} T_k) = \tag{3.19}$$
$$= H\left(\frac{p_n}{\lambda^{3+3n}}, \lambda u \right) - H_{01}\left(\frac{p_n}{\lambda^{3+3n}}, \lambda u \right) - H_{02}\left(\frac{p_n}{\lambda^{3+3n}}, \lambda u \right)$$

since

$$T_k \overset{(2.17)}{=} u^{2k+1} \sum_{n=1}^{\infty} \frac{n^{n+k}}{n!} u^{3n} p_n \tag{3.20}$$

is equivalent to

$$\lambda^{2k-2} T_k = (\lambda u)^{2k+1} \sum_{n=1}^{\infty} \frac{n^{n+k}}{n!} (\lambda u)^{3n} \frac{p_n}{\lambda^{3+3n}}. \tag{3.21}$$

It remains to put $\lambda^6 = g^2$.

3.4 Consistency with [3]

It is also easy to compare our $\mathcal{F}(T)$ with its smaller fragments, explicitly evaluated in [3] from the AMM-Eynard equation on the Lambert curve. To this end, one should interpret multidensities from that paper as

$$\rho^{(p|m)}(y_1, \ldots, y_m) = W_p(y_1, \ldots, y_m) = \hat{\nabla}(y_1) \ldots \hat{\nabla}(y_m) F^{(p)}(T) \,,$$

$$\hat{\nabla}(y) = \sum_{m=0}^{\infty} \zeta_m(y) \frac{\partial}{\partial T_m} \tag{3.22}$$

where $\zeta_m(y)$ is defined in section 5.2 below. With this interpretation it is easy to extract from eqs. (2.44)–(2.47) of [3]:

$$F_0 = \underbrace{\frac{1}{6}T_0^3 + \frac{1}{6}T_0^3 T_1 + \cdots}_{\text{genus 0}} + \underbrace{\frac{1}{24}T_1 + \frac{1}{48}T_1^2 + \frac{1}{24}T_0 T_2}_{\text{genus 1}}$$

$$+ \underbrace{\frac{1}{9 \cdot 128}T_4 + \frac{1}{9 \cdot 128}T_0 T_5 + \frac{1}{3 \cdot 128}T_1 T_4 + \frac{29}{45 \cdot 128}T_2 T_3 + \cdots}_{\text{genus 2}}$$

$$+ \underbrace{\frac{1}{9^2 \cdot 1024}T_7 + \cdots + \cdots}_{\text{genus 3}} \tag{3.23}$$

$$F_1 = \underbrace{-\frac{1}{24}T_0 - \frac{1}{24}T_0 T_1 - \cdots}_{\text{genus 1}} \underbrace{-\frac{1}{15 \cdot 32}T_3 - \frac{1}{15 \cdot 32}T_0 T_4 - \frac{1}{5 \cdot 32}T_1 T_3 - \frac{5}{9 \cdot 128}T_2^2 - \cdots}_{\text{genus 2}}$$

$$\underbrace{-\frac{7}{5 \cdot 27 \cdot 1024}T_6 - \cdots - \cdots}_{\text{genus 3}} \tag{3.24}$$

$$F_2 = \underbrace{\frac{7}{45 \cdot 128}T_2 + \frac{7}{45 \cdot 128}T_0 T_3 + \frac{7}{15 \cdot 128}T_1 T_2 + \cdots}_{\text{genus 2}} + \underbrace{\frac{41}{7 \cdot 81 \cdot 1024}T_5 + \cdots}_{\text{genus 3}} + \cdots \tag{3.25}$$

$$F_3 = \underbrace{-\frac{31}{5 \cdot 27 \cdot 7 \cdot 1024}T_4 - \cdots}_{\text{genus 3}} - \cdots \,, \tag{3.26}$$

$$\cdots$$

what obviously coincides with formulas in section 3.2 above. These formulas from [3] are written for $u = 1$, but one can easily restore the u-dependence. We return to discussion of this approach in the special section 5 below.

3.5 A few comments

1. In order to avoid possible confusion about our notation and normalization conditions we explicitly list a few first terms in the lowest Virasoro constraints:

$$\left((T_1 - 1)\frac{\partial}{\partial T_0} + \cdots \right) F_0^{(0)} = -\frac{1}{2}T_0^2 \,,$$

$$\left(T_0\frac{\partial}{\partial T_0} + 3(T_1 - 1)\frac{\partial}{\partial T_1} + \cdots\right) F_0^{(0)} = 0\,,$$

$$\left(3T_0\frac{\partial}{\partial T_1} + 15(T_1 - 1)\frac{\partial}{\partial T_2} + \cdots\right) F_0^{(0)} + \frac{1}{2}\left(\frac{\partial F_0^{(0)}}{\partial T_0}\right)^2 = 0\,,$$

$$\cdots \left((T_1 - 1)\frac{\partial}{\partial T_0} + \cdots\right) F_0^{(1)} = 0\,,$$

$$\left(T_0\frac{\partial}{\partial T_0} + 3(T_1 - 1)\frac{\partial}{\partial T_1} + \cdots\right) F_0^{(1)} = -\frac{1}{8}\,,$$

$$\left(\frac{\partial F_0^{(0)}}{\partial T_0}\frac{\partial}{\partial T_0} + 3T_0\frac{\partial}{\partial T_1} + 15(T_1 - 1)\frac{\partial}{\partial T_2} + \cdots\right) F_0^{(1)} + \frac{1}{2}\frac{\partial^2 F_0^{(0)}}{\partial T_0^2} = 0\,,$$

$$\cdots$$

$$\left((T_1 - 1)\frac{\partial}{\partial T_0} + \cdots\right) F_1^{(1)} = \frac{1}{24}\,,$$

$$\left(T_0\frac{\partial}{\partial T_0} + 3(T_1 - 1)\frac{\partial}{\partial T_1} + \cdots\right) F_1^{(1)} = -F_1^{(1)}\,, \quad (3.27)$$

$$\left(\frac{\partial F_0^{(0)}}{\partial T_0}\frac{\partial}{\partial T_0} + 3T_0\frac{\partial}{\partial T_1} + 15(T_1 - 1)\frac{\partial}{\partial T_2} + \cdots\right) F_1^{(1)} = 0\,.$$

$$\cdots$$

2. The general form of terms, explicitly shown in (3.12)–(3.16), is

$$F_q^{(p)} = \sum_k \frac{1}{(k + 2 + q - 3p)!}\frac{T_0^{k+2+q-3p}T_k}{(1 - T_1)^{k+1+q-p}} \qquad (3.28)$$

$$+ \sum_k \frac{(k + 1 + q - p)(k + 2 + q - p) + \beta_q^{(p)}}{2(k + 3 + q - 3p)!}\frac{T_0^{k+3+q-3p}T_2T_k}{(1 - T_1)^{k+3+q-p}} + \cdots\,.$$

The values of $\beta_q^{(p)}$ are not constrained by $\hat{L}_0$ and $\hat{L}_1$ conditions: these values are examples of γ-parameters.

3. It is interesting to note that, if all β are vanishing, one would have

$$F_q^{(p+q)} \approx \partial_0^{2q} F_0^{(p)}\,. \qquad (3.29)$$

In order to understand this, note that, using (2.15), one obtains

$$\hat{l}_{-1}F_q^{(p)} = \frac{\delta_{q,1}\delta_{p,1}}{24} - \frac{T_0^2}{2}\delta_{p,0}\delta_{q,0} \qquad (3.30)$$

and, therefore,

$$\hat{l}_{-1}\partial_0^2 F_q^{(p)} = -\delta_{p,0}\delta_{q,0}\,. \qquad (3.31)$$

Similarly, using (3.1), one obtains

$$(\hat{l}_0 + q)F_q^{(p)} = -\frac{\delta_{q,0}\delta_{p,1}}{16} \qquad (3.32)$$

and, therefore,

$$(\hat{l}_0 + q + 1)\partial_0^2 F_q^{(p)} = 0 \,. \tag{3.33}$$

For $q, p > 1$ this means that $F_{q+1}^{(p+1)}$ and $\partial_0^2 F_q^{(p)}$ satisfy the same first two Virasoro constraints and the same selection rules (otherwise one could take $F_{q+1}^{(k)}$ with any $k > 1$, since it satisfies the same two Virasoro constraints). This does not mean that these two functions coincide, since there are higher constraints, which fix the ambiguity expressed in terms of arbitrary coefficients γ. For instance, in most cases $\beta \neq 0$, and (3.29) acquires corrections (i.e. the γ-parameters are all different). However, in those cases when there is no freedom in solutions of the two first constraints (there are no many γ's), relation (3.29) is correct. For instance,

$$F_2^{(2)} \sim \partial_0^2 F_1^{(1)} \tag{3.34}$$

since, as it follows from (3.13), (3.14), there is no γ-freedom in these free energies but the general coefficient.

Moreover, even the relative coefficient is fixed in the combination $F_1^{(1)} + \frac{1}{24}\frac{\partial^2}{\partial T_0^2}F_0^{(0)}$ which is canceled by the first two Virasoro constraints (see (3.30)–(3.33)), which means that

$$F_1^{(1)} = -\frac{1}{24}\frac{\partial^2}{\partial T_0^2}F_0^{(0)} \,. \tag{3.35}$$

Thus, we provided a decisive evidence that the Kontsevich-Hurwitz partition function is, indeed, given by (2.10), i.e. is a solution to the conjugated Virasoro constraints (2.9). This means that it is one of the phases in the M-theory of matrix models [5]. From here, the reader can directly proceed to our conclusions in section 6. Still we find the claims of $[1-3]$ so interesting, that we devote the next two sections section 4 and section 5 to deeper discussion about the claims of these papers.

4. Hurwitz partition function $H(p)$

We do not go into details of this very interesting story, which is nicely presented in numerous papers. Only some facts of direct relevance for our consideration are briefly reviewed in this section.

4.1 Hurwitz numbers

Hurwitz numbers count ramified coverings of a Riemann sphere. Relevant for our considerations are coverings with N sheets, connected only pairwise (double ramifications) except for at a single point (usually posed at infinity), where one can glue together $m_1, m_2, \ldots, m_n$ sheets, with $\sum_{i=1}^n m_i = N$ and some $m_i \geq 1$. The number of double ramification (i.e. of simple critical) points is then equal to

$$M = 2p - 2 + \sum_{i=1}^n (m_i + 1) \tag{4.1}$$
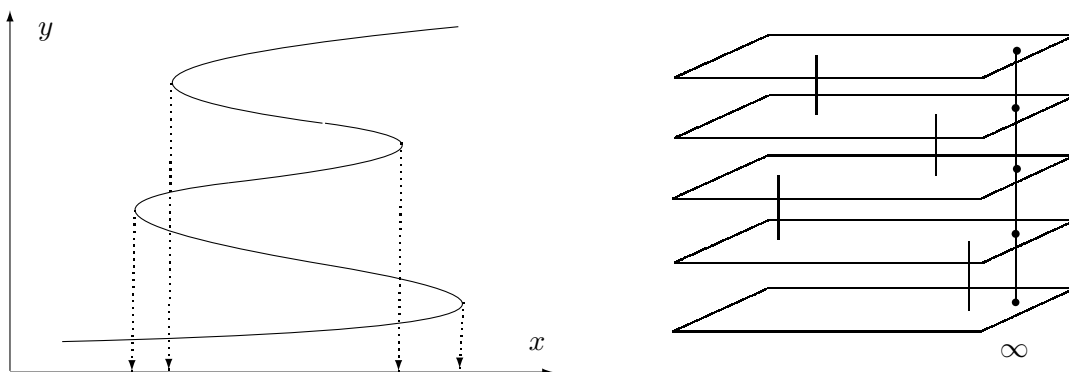
Figure 1: The covering $y \to x$ of the Riemann sphere in the simplest case of the curve $Q_N(y) = x$. Left picture: the real section. Right picture: symbolical complex view. All critical points (zeroes of $Q'(y)$ are assumed different. The N sheets merge together at infinity.

where p is the genus of the covering. Positions of ramification points (moduli) are not taken into account, only combinatorics.

Figure 1 illustrates the setting in the simplest possible case of the covering, $y \to x$ described by the equation $Q_N(y) = x$, Q_N being a polynomial of degree N. The function $y(x)$ has N branches, and its Riemann surface is an N-fold covering of the Riemann sphere, parameterized by x. The covering is ramified at $N-1$ zeroes of the derivative $Q'_N(y_i) = 0$, i.e. at $x_i = Q_N(y_i)$, which are all assumed different (condition that the critical/ramification points are simple/double) and at $x = \infty$, where all the N sheets of the Riemann surface are glued together. Thus, in this case $n = 1$, $m_1 = N$, $M = N - 1$ and, obviously, $p = 0$ — in accordance with the Riemann-Hurwitz formula (4.1).

Another familiar case is the hyperelliptic covering $y^2 = P_{2p+1}(x)$ where all the ramification points, including the one at infinity are simple (double). In this case $n = 1$, $m_1 = 2$ and $M = 2p + 1$.

In the opposite extreme case of *generic* irreducible polynomial of degree N, $P_N(x, y) = y^N + \sum_{k+l \leq N} y^k x^l = 0$, the function $y(x)$ has N branches, i.e. its Riemann surface has N sheets of the corresponding and is ramified at $N(N-1)$ points, where $\frac{\partial P_N}{\partial y} = 0$ (discriminant $\mathrm{Disc}_y P_n(x, y)$ is a polynomial of degree $N(N-1)$ in x), which are (generically) all different, and at $x = y = \infty$. The branching at infinity is controlled by the *homogeneous* part of $P_N(x, y)$ (a "symbol" of P_N), which is fully reducible, $P_N(x, y) \sim y^N + \sum_{k+l=N} p_{kl} y^k x^l = \prod_{i=1}^N (y - \lambda_i x)$ as $x, y \to \infty$. This means that there is actually no branching at infinity, thus $n = N$, $m_1 = \cdots = m_N = 1$ and genus $p = \frac{(N-1)(N-2)}{2}$.

However, if our polynomial has different degrees N and $n \leq N$ in y and x respectively and behaves as $P(x, y) \sim \prod_{i=1}^n (y^{m_i} - \lambda_i x)$ at $x, y \to \infty$ then things are different: non-trivial branching structure occurs over $x = \infty$, and it is characterized by partition $N = \sum_{i=1}^n m_i$ of N, see figure 2 for a simple example. However, partition does not characterize the
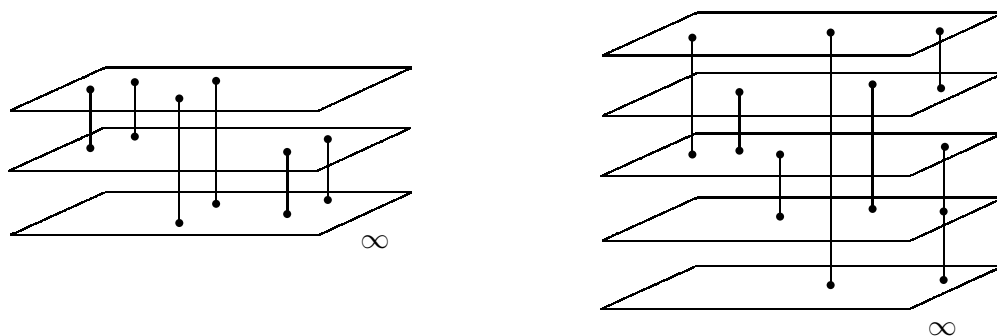
Figure 2: The covering $y \to x$ of the Riemann sphere in the case of generic $P_N(x,y) = 0$. Left picture: a fully reducible symbol, no branching at infinity. Actually in the picture $N = 3$ and $M = 6$, so that $p = 1$ (this is the cubic representation of a torus, like $x^3 + y^3 + \alpha xy = 0$). Right picture: generic branching at infinity, with n groups of merging $m_1, m_2, \ldots, m_n$ sheets. Actually in the picture $n = 2$, $m_1 = 2$, $m_2 = 3$, $M = 5$, $p = 0$.

covering unambiguously: what remains not fixed, is combinatorics of pairwise gluing of sheets, and Hurwitz number $h(p|m_1, \ldots, m_n)$ counts the number of different possibilities (modulo *location* of the critical points — if they are taken into account we get a whole continuous moduli space of coverings and Hurwitz number counts the number of *its* sheets).

Hurwitz numbers are simple to define, but not so easy to calculate. As usual, the problem is drastically simplified by passing to generating functions — this is one of the ideas, put into the basis of string theory. Moreover, not a single, but a number of various (dual) descriptions immediately arise in this way. Most straightforwardly, the Hurwitz free energy is the generating function

$$H(p) = \frac{1}{g^2} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{p; m_1, \ldots, m_n; M} \delta \left(\sum_{i=1}^{n} (m_i + 1) + 2p - 2 - M \right) \frac{u^{3M} g^{2p}}{M!} h(p \,|\, m_1, \ldots, m_n) p_{m_1} \ldots p_{m_n}$$

(4.2)

The parameters u and g serve to separate Hurwitz numbers for different numbers M of simple ramification points and different genera. As already mentioned in (3.20), eq. (4.1) allows one to absorb g^2 into rescalings of u and p_m: $u \to gu$, $p_m \to p_m/g^{m+1}$, and we do not keep g^2 dependence explicitly in this section.

4.2 e^H as KP τ-function

According to [18], the exponential of $H(p|u)$ can be alternatively represented by eq. (2.4):

$$e^{H(p)} = e^{u^3 \hat{W}_0(p)} e^{p_1}$$

(4.3)

what immediately implies that it is a KP τ-function [19], simply because e^{p_1} is, and all the W_∞ generators belong to $GL(\infty)$ which is the symmetry group of the Universal Grassmannian [29, 30]. Eq. (4.3) is motivated by relation to partitions and characters, see
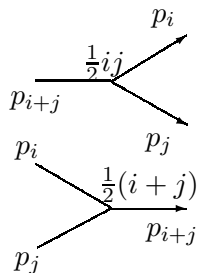
Figure 3: The two vertices in diagram technique which describes the action of $\hat{W}_0$ on e^{p_1}. Arrows denote derivatives with respect to p, ends without arrows carry p themselves. Vertices contain factors of ij and $i + j$. In what follows we often write i instead of p_i.

also [31, 32], but comments on its derivation are beyond the scope of this paper. We concentrate instead on its implications.

According to (4.3), $H(p)$ is obtained by the diagram technique with two triple-vertex elements (similar to the one analyzed in [33]), see the appendix and figure 3. Direction of arrows is important, lines with different orientation are different, since the weights of two vertices, ij and $i + j$ do not coincide, i.e. these are rather Heitler than Feynman diagrams. All possible diagrams describe the r.h.s. of (4.3), and its logarithm, $H(p)$ contains only *connected* diagrams. The power of u^3 is the number of vertices, however, this does not immediately provide the power of u in $\mathcal{F}(T)$, because an u-dependence is also contained in $T(p)$ (moreover, the diagrams with odd number of vertices contain odd powers of u, which are all converted into even powers after the transformation from p to T). Remarkably, despite q does not have a direct diagrammatic meaning, p has: the diagram with p loops contributes only to components $F_q^{(p)}$ of the free energy. H_{01} is the sum of all rooted tree diagrams, and H_{02} of all double-rooted trees. The detailed account of this diagram technique can be found in the appendix.

4.3 The claims of [1, 2]

4.3.1 Relation between Hurwitz and Kontsevich-Hodge free energies

The Hurwitz free energy (4.2), which is the generating function of the Hurwitz numbers and the Kontsevich-Hodge free energy (2.2), which is the generating function of the Hodge integrals, can be related with the help of the ELSV formula [16]:

$$\frac{1}{M!}h(p|m_1,\ldots,m_n) = \prod_{i=1}^n \frac{m_i^{m_i}}{m_i!} \int_{\mathcal{M}_{p,n}} \frac{1 - \lambda_1 + \lambda_2 - \cdots \pm \lambda_p}{\prod_{i=1}^n (1 - m_i\psi_i)}, \qquad (4.4)$$

$$M = 2p - 2 + \sum_{i=1}^n (m_i + 1).$$

The Hodge integrals

$$I_q^{(p)}(k_1,\ldots,k_n) = \int_{\mathcal{M}_{p,n}} \lambda_q \prod_{i=1}^n \psi_i^{k_i} \qquad (4.5)$$

do not vanish only provided

$$q + \sum_{i=0}^{n}(k_i - 1) = 3p - 3.\tag{4.6}$$

This implies [2] that, when (4.5) is multiplied by u^{3m}, the u-factors can be redistributed as follows:

$$\frac{u^{3M}}{M!}h(p\,|\,m_1,\ldots,m_n) = \prod_{i=1}^{n}\frac{m_i^{m_i}}{m_i!}u^{3m_i+1}\int_{\mathcal{M}_{p,n}}\frac{1 - u^2\lambda_1 + u^4\lambda_2 - \cdots \pm u^{2p}\lambda_p}{\prod_{i=1}^{n}(1 - m_i\psi_i u^2)}.\tag{4.7}$$

Indeed, the total power of u in the integral is $u^{2(j+\sum k_i)} = u^{6p-6+2n}$. Together with $\prod_i u^{3m_i+1}$ this gives $u^{6p-6+3\sum(m_i+1)} = u^{3M}$, as required.

Converting (4.7) with p-variables, one obtains [2]:

$$H(p) \overset{(4.2)}{=} \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{p;m_1,\ldots,m_n;M}\frac{u^{3M}}{M!}h(p\,|\,m_1,\ldots,m_n)p_{m_1}\ldots p_{m_n}\delta$$

$$\left(\sum_{i=1}^{n}(m_i+1) + 2p - 2 - M\right)\overset{(4.7)}{=}$$

$$= \sum_{p,n=0}^{\infty}\frac{1}{n!}\int_{\mathcal{M}_{p,n}}\left(1 - u^2\lambda_1 + u^4\lambda_2 - \cdots \pm u^{2p}\lambda_p\right)\prod_{i=1}^{n}\left(\sum_{m_i=1}^{\infty}\frac{m_i^{m_i}}{m_i!}\frac{u^{3m_i+1}p_{m_i}}{(1 - m_i\psi_i u^2)}\right) =$$

$$= \sum_{q=0}^{\infty}(-)^q u^{2q}\sum_{n=0}^{\infty}\frac{1}{n!}\int_{\mathcal{M}_{p,n}}\lambda_q\prod_{i=1}^{n}\left(\sum_{k=0}^{\infty}T_k\psi_i^k\right) =$$

$$= \sum_{q=0}^{\infty}(-)^q u^{2q}\sum_{p,n=0}^{\infty}\frac{1}{n!}\delta\left(\sum_{i=1}^{n}(k_i-1) - (3p-3-q)\right)I_q^{(p)}(k_1,\ldots,k_n)\,T_{k_1}\ldots T_{k_n}$$

$$= \sum_{p\geq q\geq 0}u^{2q}F_q^{(p)}(T)\tag{4.8}$$

with

$$T_k = u^{2k+1}\sum_{m\geq 1}^{\infty}\frac{m^{m+k}}{m!}u^{3m}p_m.\tag{4.9}$$

This explains the relation (2.17) between T and p variables. The factors g^{2p} are introduced straightforwardly, as was already demonstrated in section 3.3.

The restriction $p \geq q$ is an important property of the Hodge integrals. It is one of non-trivial things to be checked in analyzing our main claim (2.10).

Interrelations between different objects/quantities that emerge when dealing with Hurwitz numbers and Hodge integrals are summarized in table 1:

	Hurwitz	Hodge
curve	a ramified covering of Riemann sphere with all but one (at ∞) critical points simple	arbitrary curve
p	genus of the covering	genus of the curve
n	number of different preimages of ∞	number of marked points on a complex curve
$\{m_1, \ldots, m_n\}$	multiplicities of different preimages	—
M	number of simple (double) ramification points	—
ψ_i	—	Chern$_1$ of the bundle of cotangent lines at i-th marked point
λ_j	—	Chern$_j$ of the bundle of holomorphic 1-forms

Table 1: Hurwitz numbers vs. Hodge integrals.

The r.h.s. of eq. (4.8) can be also rewritten in terms of Witten's topological correlators [34][5]

$$I_q^{(p)} = \int_{\mathcal{M}_{p,n}} \lambda_q \prod_{i=1}^{n} \psi_i^{k_i} = \left\langle \lambda_q \sigma_{k_1} \ldots \sigma_{k_n} \right\rangle, \qquad \sum_{i=0}^{n}(k_i - 1) = 3p - 3 - q \qquad (4.10)$$

In this notation

$$\mathcal{F} = \sum_{q,m_0,m_1,\ldots} u^{2q} \left\langle \lambda_q \sigma_0^{m_1} \sigma_1^{m_1} \ldots \right\rangle \frac{T_0^{m_0}}{m_0!} \frac{T_1^{m_1}}{m_1!} \ldots = \sum_{q=0}^{\infty} u^{2q} \left\langle \lambda_q \exp\left(\sum_{k=0}^{\infty} T_k \sigma_k\right) \right\rangle. \qquad (4.11)$$

4.3.2 Interplay between the T, p and q time-variables

We can now elaborate more on the $T - p$ relation (4.9). First of all, one can rewrite it recursively and, in terms of the generating functions,

$$T_k = u^{2k+4} \sum_{n=1}^{\infty} \frac{n^{n+k}}{n!} u^{3n} p_n = u^{2k+1} \left(\frac{u}{3}\frac{\partial}{\partial u}\right)^k \left(\frac{T_0}{u}\right) \qquad (4.12)$$

where

$$\frac{1}{u} T_0 = \sum_{n=1}^{\infty} \frac{n^n}{n!} u^{3n} p_n = \oint w(x) dp(x) \qquad (4.13)$$

with

$$dp(x) \equiv \sum_n \frac{p_n dx}{x^{n+1}} \qquad (4.14)$$

and

$$w(x) \equiv \sum_{n=1}^{\infty} \frac{n^n}{n!} (u^3 x)^n \qquad (4.15)$$

---

[5]This definition, in accordance with [34], implies that adding σ_k in the correlator corresponds to ψ^k inserted at any point, i.e. σ_k^l just counts the number of the corresponding operators inserted, irrespectively on their location. This is a correct definition, since in topological theory correlators do not depend on locations of operators.

is an inverse of the function

$$u^3 x = \frac{w}{1+w} \exp\left(-\frac{w}{1+w}\right) \overset{(A.22)}{=} w_{-1} \exp\left(-w_{-1}\right). \tag{4.16}$$

$T(p)$ is linear, but not a triangular change of time-variables. It can be decomposed into upper and lower triangular transformations.

Expanding powers of x into formal series in w, we introduce a set of expansion coefficients $c_{n|k}$:

$$(u^3 x)^n = \sum_{k \geq n} c_{n|k} w^k. \tag{4.17}$$

Then

$$p_n = \oint x^n dp(x) = \sum_{k \geq n} c_{n|k} u^{3(k-n)} q_k \tag{4.18}$$

where

$$q_k = u^{-3k} \oint w^k(x) dp(x). \tag{4.19}$$

We can now express the T-variables through the q-variables. Obviously,

$$T_0 = u \oint w(x) dp(x) = u^4 q_1. \tag{4.20}$$

Next,

$$T_1 = \frac{u^4}{3} \oint \frac{\partial w(x)}{\partial u} dp(x). \tag{4.21}$$

The derivative of w is taken at constant x and can be obtained by differentiating (4.16),

$$u^3 x = \left(\frac{u}{3} \frac{\partial w(x)}{\partial u}\right) \frac{d}{dw} \left\{\frac{w \exp\left(-\frac{w}{1+w}\right)}{1+w}\right\} = \frac{\exp\left(-\frac{w}{1+w}\right)}{(1+w)^3} \left(\frac{u}{3} \frac{\partial w(x)}{\partial u}\right). \tag{4.22}$$

Substituting (4.16) at the l.h.s., one gets

$$\frac{u}{3} \frac{\partial w(x)}{\partial u} = w(1+w)^2 \tag{4.23}$$

and

$$T_1 = u^3 \oint w(1+w)^2 dp = u^6 q_1 + 2u^9 q_2 + u^{12} q_3. \tag{4.24}$$

Next,

$$T_2 = u^5 \frac{u}{3} \frac{\partial}{\partial u} \frac{T_1}{u^3} = u^5 \frac{u}{3} \frac{\partial}{\partial u} \oint w(1+w)^2 dp = u^5 \oint w(1+w)^2 (1+4w+3w^2) dp =$$
$$= u^8 q_1 + 6u^{11} q_2 + 12u^{14} q_3 + 10u^{17} q_4 + 3u^{20} q_5. \tag{4.25}$$

In the same way one can deduce expressions for all other

$$T_k = u^{2k+1} \oint w_k(w)dp, \quad w_{k+1} \overset{(A.20)}{=} w(1+w)^2 \frac{dw_k}{dw}. \quad (4.26)$$

They describe a triangular change of variables $T(q)$, with T_k being a linear combination of $q_1, \ldots, q_{2k+1}$, and the recurrent relation (4.26) can be immediately rewritten in terms of q's [2]:

$$T_{k+1} = u^2 \sum_{m=1}^{2k+1} m(q_m + 2u^3 q_{m+1} + u^6 q_{m+2}) \frac{\partial T_k}{\partial q_m}. \quad (4.27)$$

4.3.3 Comparing H and $\mathcal{F}$ expressed through the q-variables

A few lowest transformations $p(q)$ and $T(q)$ look as follows:

$$
\begin{aligned}
p_1 &= q_1 - 2u^3 q_2 + \frac{7}{2} u^6 q_3 - \frac{17}{3} u^9 q_4 + \frac{209}{24} u^{12} q_5 - \cdots, \\
p_2 &= q_2 - 4u^3 q_3 + 11 u^6 q_4 - \frac{76}{3} u^9 q_5 + \cdots, \\
p_3 &= q_3 - 6u^3 q_4 + \frac{45}{2} u^6 q_5 - \cdots, \\
p_4 &= q_4 - 8u^3 q_5 + \cdots, \\
p_5 &= q_5 + \cdots, \\
&\quad \cdots
\end{aligned}
\quad (4.28)
$$

and

$$
\begin{aligned}
T_0 &= u^4 \Big(p_1 + 2u^3 p_2 + \frac{9}{2} u^6 p_3 + \frac{32}{3} u^9 p_4 + \cdots \Big) = u^4 q_1, \\
T_1 &= u^6 \Big(p_1 + 4u^3 p_2 + \frac{27}{2} u^6 p_3 + \frac{128}{3} u^9 p_4 + \cdots \Big) = u^6 q_1 + 2u^9 q_2 + u^{12} q_3, \\
T_2 &= u^8 \Big(p_1 + 8u^3 p_2 + \frac{81}{2} u^6 p_3 + \frac{512}{3} u^9 p_4 + \cdots \Big) \\
&= u^8 q_1 + 6u^{11} q_2 + 12 u^{14} q_3 + 10 u^{17} q_4 + 3 u^{20} q_5, \\
T_3 &= u^{10} \Big(p_1 + 16 u^3 p_2 + \frac{243}{2} u^6 p_3 + \frac{2048}{3} u^9 p_4 + \cdots \Big) \\
&= u^{10} q_1 + 14 u^{13} q_2 + 61\, u^{16} q_3 + 124 u^{19} q_4 + 131\, u^{22} q_5 + 70 u^{25} q_6 + 15 u^{28} q_7, \\
&\quad \cdots
\end{aligned}
$$

Note once again that T_m are infinite series in terms of p, but are *finite* linear combinations of q's.

Given these expressions, one can substitute them into the Hurwitz and Kontsevich-Hurwitz free energies:

$$
\begin{aligned}
\mathcal{H}(q) = H(p)|_{p \to p(q)} &= \frac{1}{12} u^9 q_2 + \frac{1}{48} u^{12} (2q_3 + q_1^2 + 8q_1^3) + \frac{1}{240} u^{15} (q_2 + 60 q_1 q_2) \\
&\quad + \frac{1}{1440} u^{18} (120 q_2^2 + 138 q_3 + 80 q_1^3 + q_1^2 + 720 q_1 q_3 + 240 q_1^4) + O(u^{21})
\end{aligned}
\quad (4.29)
$$

while

$$\mathcal{F}(u|q) \equiv \sum_{n,l} \frac{u^{2n}}{l!} \sum_{k_i \geq 0} F_{n,k_1 k_2 \ldots k_l} T_{k_1} T_{k_2} \ldots T_{k_l} =$$

$$= F_{0,0} u^4 q_1 + \underline{F_{0,1}}(u^6 q_1 + 2u^9 q_2 + u^{12} q_2) + F_{0,2} u^8 q_1 + \cdots$$

$$+ \frac{1}{2} F_{0,00} u^8 q_1^2 + F_{0,01} u^{10} q_1(q_1 + 2u^3 q_2 + u^6 q_2) + \cdots \qquad (4.30)$$

$$+ \frac{1}{6} \underline{F_{0,000}} u^{12} q_1^3 + \cdots$$

$$+ u^2 \underline{F_{1,0}} u^4 q_1 + \cdots .$$

In the last formula we expanded the free energy $\mathcal{F}(T)$ into Taylor series in T and made a substitution $T \to T(q)$. It is important that many terms are actually absent in Taylor expansions, because the corresponding derivatives of $F_q^{(p)}$ are vanishing. Underlined in the above formula are the terms with non-vanishing F-derivatives. Thus

$$F_0 = \left\langle e^{T_k \sigma_k} \right\rangle = \underbrace{\frac{1}{6} T_0^3 + \frac{1}{6} T_0^3 T_1 + \frac{1}{6} T_0^3 T_1^2 + \frac{1}{24} T_0^4 T_2 + \cdots}_{\text{genus } 0}$$

$$+ \underbrace{\frac{1}{24} T_1 + \frac{1}{48} T_1^2 + \frac{1}{24} T_0 T_2 + \frac{1}{72} T_1^3 + \frac{1}{12} T_0 T_1 T_2 + \frac{1}{48} T_0^2 T_3 + \cdots}_{\text{genus } 1}$$

$$+ \underbrace{\frac{1}{1152} T_4 + \frac{29}{5760} T_2 T_3 + \frac{1}{384} T_1 T_4 + \frac{1}{1152} T_0 T_5 + \cdots}_{\text{genus } 2} + \underbrace{\cdots}_{\text{higer genera}} =$$

$$= \frac{1}{24} u^6 q_1 + \frac{1}{12} u^9 q_2 + u^{12} \left(\frac{1}{6} q_1^3 + \frac{1}{16} q_1^2 + \frac{1}{1152} q_1 + \frac{1}{24} q_3 \right) + \cdots, \qquad (4.31)$$

$$F_1 = \left\langle \lambda_1 e^{T_k \sigma_k} \right\rangle = \underbrace{-\frac{1}{24} T_0 - \frac{1}{24} T_0 T_1 - \cdots}_{\text{genus } 1} \underbrace{- \frac{1}{15 \cdot 32} T_3 + \cdots}_{\text{genus } 2} + \underbrace{\cdots}_{\text{higer genera}} \rightarrow$$

$$\xrightarrow{\times u^2} -\frac{1}{24} u^6 q_1 - \frac{1}{24} u^{12} q_1^2 - \frac{1}{15 \cdot 32} u^{12} q_1 - \cdots, \qquad (4.32)$$

$$F_2 = \left\langle \lambda_2 e^{T_k \sigma_k} \right\rangle = \underbrace{\frac{7}{45 \cdot 128} T_2 + \cdots}_{\text{genus } 2} + \underbrace{\cdots}_{\text{higer genera}} \rightarrow$$

$$\xrightarrow{\times u^4} \frac{7}{45 \cdot 128} u^{12} q_1 + \cdots, \qquad (4.33)$$

Therefore, one has

$$\mathcal{F}(u|q) = \frac{1}{12} u^9 q_2 + \frac{1}{48} u^{12}(2q_3 + q_1^2 + 8q_1^3) + O(u^{15}) = \frac{1}{12} u^9 q_2 + O(u^{12}), \qquad (4.34)$$

in agreement with (4.29).

Note that the agreement is based on non-trivial relations between different components $F_q^{(p)}$ like $F_{1,0} = -F_{0,1}$ (to cancel the u^6 terms) or $\frac{1}{1152} - \frac{1}{15 \cdot 32} + \frac{7}{45 \cdot 128} = 0$ etc.

4.3.4 Kontsevich-Hurwitz partition function as a KP τ-function: the need to switch from T to q

Substituting $q_m = u^{-4m} \check{q}_m$ and taking the limit $u \to 0$, one gets

$$T_k \overset{u \to 0}{\longrightarrow} (2k+1)!! \, \check{q}_{2k+1} \qquad (4.35)$$

so that one can identify $2^k \check{q}_{2k+1}$ with τ_k in the Kontsevich model. Since τ_k are time-variables of the KdV τ-function, $\check{q}_k$ provide a doubled set of variables, natural for description of KP τ-function. Indeed, as proved in [2], if expressed through the q-variables, $\mathcal{Z}\left(T(q)\right)$ becomes a KP τ-function.

To derive this claim, one has to start from the obvious fact that $\exp\left(H(p)\right)$ is a KP τ-function in p-variables – simply because it is obtained by the action of a W-operator from a trivial τ-function e^{p_1} and all the generators of W_∞-algebra belong to $GL(\infty)$, which acts on the universal Grassmannian, considered as a universal moduli space of Riemann surfaces [29, 35], and maps KP solutions into KP solutions. This, however, does not imply that $\mathcal{Z}\left(T(p)\right)$ is a KP τ-function, since the KP hierarchy is not invariant w.r.t a generic change of variables. Moreover, $\mathcal{F}\left(T(p)\right)$ coincides not with $H(p)$ but with $H(p) - H_{01}(p) - H_{02}(p)$, and subtraction of the quadratic function H_{02} would also violate the KP equations.

However, would the change of times be induced by a change of the spectral parameter, one still can obtain a KP τ-function. Indeed, as follows from the theory of equivalent hierarchies [22–24], if one makes a change of the spectral parameter $\mu \to \tilde{\mu}(\mu) = \mu + \sum_{k \geq 0} \mu^{-k}$ at the vicinity of infinity, the times are changed by the following triangle transformation [23, eq.(16)]

$$k \tilde{t}_k = \sum_l \mathrm{Res}_{\mu = \infty} \frac{\mu^{l-1}}{\tilde{\mu}^k(\mu)} l t_l \qquad (4.36)$$

while the τ-function in $\tilde{t}$-variables is multiplied by the exponential quadratic in times [23, eq. (46)]:

$$\tau(t) = \exp\left(-\frac{1}{2} \sum_{kl} Q_{kl} \tilde{t}_k \tilde{t}_l\right) \times \tilde{\tau}(\tilde{t})$$
$$Q_{kl} = \mathrm{Res}_{\mu = \infty}\left\{\tilde{\mu}^k(\mu) \frac{d\left[\tilde{\mu}^l(\mu)\right]_+}{d\mu}\right\} \qquad (4.37)$$

where $[\ldots]_+$ denotes only positive powers of the power series.

As emphasized in [2], the change of time variables $p \to q$ is exactly of this type. Indeed, upon identification of $1/w$ and $1/x$ related by formula (4.16) with μ and $\tilde{\mu}$ respectively, one immediately reproduces from (4.36) the change of time variables (4.28) and from (4.37) the proper $Q = H_{02}$ in (2.18), the times being identified as $q_k = k t_k$, $p_k = k \tilde{t}_k$. Note that the linear part H_{01} of (2.18) is not reproduced by integrability theory arguments, since the KP τ-function is defined up to an arbitrary exponential factor linear in times. Note that, although this is the invariance of the Hirota bilinear equations, if one would like to preserve some reduction of the KP hierarchy, it is necessary to choose this exponential properly.

From the point of view of the conjugated Virasoro algebra (2.9) the need to switch from T to q-variables in order to obtain a KP τ-function is related to the fact that operator $\hat{N}_1$ does *not* belong to the $GL(\infty)$ algebra, which acts on the universal Grassmannian [29].

5. Conjugated Virasoro constraints (2.9) in the BM approach

5.1 AMM-Eynard equations

AMM-Eynard equations rewrite Virasoro constraints in terms of a spectral complex curve Σ in the following form

$$\left\{ \left(V' + g^2\hat{\nabla}\right) * \left(V' + g^2\hat{\nabla}\right) \right\} Z = 0 \, . \tag{5.1}$$

Later on in this section, we put $g = 1$. Here

$$\hat{\nabla}(z) = \sum_{k=0}^{\infty} \zeta_k(z) \frac{\partial}{\partial T_k} \tag{5.2}$$

$$V'(z) = \sum_{k=0}^{\infty} \tilde{T}_k v_k(z) \tag{5.3}$$

where $v_k(z)$ and $\zeta_k(z)$ are the full sets of 1-forms on Σ, related by the condition

$$\hat{\nabla}(z)V'(z') = B(z, z') \tag{5.4}$$

where $B(z, z')$ is the Bergmann kernel, i.e. $(1,1)$ Green function $B(z, z') = <\partial\phi(z)\partial\phi(z')>$ on Σ. The star product in (5.1) denotes a multiplication map $\Omega_\Sigma \times \Omega_\Sigma \to \Omega_\Sigma$ on the space of 1-forms Ω_Σ,

$$(\omega_1 * \omega_2)(z) = \sum_i \oint_{a_i} K(z, z')\omega_1(z')\omega_2(\tilde{z}') \tag{5.5}$$

which represents the projection on the "$-$" part of the Virasoro algebra. For hyperelliptic curves, which are double coverings of the Riemann sphere, $\{a_i\}$ is a finite set of ramification points and $\tilde{z}$ is the counterpart of z on the other sheet. Then the kernel K is actually a differential of the form $\frac{dz}{d\tilde{z}'}$, which is a ratio of the $(1,0)$ Green function on Σ (which is the primitive of the Bergmann kernel w.r.t. the second argument calculated from z' to $\tilde{z}'$) and the Seiberg-Witten-Dijkraaf-Vafa differential:[6]

$$K(z, z') = \frac{<\partial\phi(z)\,\phi(z')> - <\partial\phi(z)\,\phi(\tilde{z}')>}{\Omega_{\text{DV}}(z') - \Omega_{\text{SW}}(\tilde{z}')} \, . \tag{5.6}$$

See [5] and [36, 7] for details.

---

[6]In simplest case of the sphere, $\Sigma_H : y_H^2(z) = z^2 - 4S$ corresponding to the Hermitean one-matrix model [5]

$$K(z, z') = \frac{dz}{dz'} \frac{1}{z - z'} \left(\frac{1}{y_H(z)} - \frac{1}{y_H(z')} \right) \, .$$

Substitution of (5.2) and (5.3) into (5.1) gives:

$$\left(\sum_{k,n\geq 0} (v_k * \zeta_n) \tilde{T}_k \frac{\partial}{\partial T_n} + \frac{1}{2} \sum_{k,l\geq 0} (\zeta_k * \zeta_l) \frac{\partial^2}{\partial T_k \partial T_l} + \frac{1}{2} \sum_{k,l\geq 0} (v_k * v_l) \tilde{T}_k \tilde{T}_l + \frac{1}{2} Tr_* B \right) Z = 0$$

$$(5.7)$$

where, following B. Eynard, we have corrected (5.1) by adding the $*$-trace of the Bergmann kernel (note that this prescription is more than just normal ordering of $:V\hat{\nabla}:$). Expanding the products of 1-forms into linear combinations of ζ (no v will arise due to projection property of the $*$-product), one obtains a one-dimensional set of constraints on Z. They can be also written as recurrent relations for the multiresolvents

$$\rho^{(p|m)}(z_1, \ldots, z_m) = \hat{\nabla}(z_1) \ldots \hat{\nabla}(z_m) F^{(p)} \Big|_{T_k = \delta_{k,1}}$$

$$(5.8)$$

in the following form

$$\rho^{(p|m+1)}(z, z_1, \ldots, z_m) = \frac{1}{2} Tr_* B(\bullet, \bullet) \delta_{p,0} \delta_{m,0} + \sum_{i=1}^{m} B(\bullet, z_i) * \rho^{(p|m)}(\bullet, z_{I/i})$$

$$+ \sum_{p_1=0}^{p} \sum_{J \subset I} \rho^{(p_1|m_J+1)}(\bullet, z_J) * \rho^{(p_2|m_{I/J}+1)}(\bullet, I/J)$$

$$+ \frac{1}{2} Tr_* \rho^{(p-1|m+2)}(\bullet, \bullet, z_1, \ldots, z_m).$$

$$(5.9)$$

They are obtained simply by acting with operators $\hat{\nabla}$ on (5.1) or (5.7) and putting $T_k - \delta_{k,1} = 0$ afterwards. The terms with the Bergmann kernel come from the action of $\hat{\nabla}$ on V', action on the $V' * V'$ term gives rise to the trace of the Bergmann kernel. The notation here is hopefully obvious: the bullets, $\bullet$ mark arguments on which the $*$-product acts, two points are converted into a single z. If both bullets are arguments in the same function, we call the corresponding product $*$-trace, Tr_*: for, say, $H(z_1, z_2) = \sum_{m,n} H_{mn} \zeta_m(z_1) \zeta_n(z_2)$ the $*$-trace is $Tr_* H(\bullet, \bullet) = \sum_{m,n} H_{mn}(\zeta_m * \zeta_n)(z)$.

5.2 *-calculus on Lambert curve

Bouchard and Marino [3] suggested to represent the Virasoro constraints for the Kontsevich-Hurwitz partition function in the form of the AMM-Eynard equation with the Lambert curve $x = (z+1)e^{-z}$ with ramification point at $(x, z) = (1, 0)$ in the role of Σ with

$$B(z, z') = \frac{dz dz'}{(z - z')^2}$$

$$(5.10)$$

and $K(z, z')$ given by the 5-dimensional Seiberg-Witten-Dijkgraaf-Vafa differential[7] $\Omega_{SW} = -\log y \, d\log x = -\log(1 + z) d\log x$:

$$K(z, z') = \frac{\frac{dz}{z - z'} - \frac{dz}{z - \tilde{z}'}}{\log(1 + z') - \log(1 + \tilde{z}')} \frac{1 + z'}{z' dz'}.$$

$$(5.11)$$

---

[7]To compare with [3] note that our z is the same as in that paper (i.e. $z = y - 1$), but x differs by a factor of e.

Accordingly

$$\tilde{z} = S(z) = -z + \frac{2}{3}z^2 - \frac{4}{9}z^3 + \frac{44}{135}z^4 - \frac{104}{405}z^5 + \frac{40}{189}z^6 - \frac{7648}{42525}z^7 + \frac{2848}{18225}z^8$$
$$-\frac{31712}{229635}z^9 + + \frac{23429344}{189448875}z^{10} - \frac{89072576}{795685275}z^{11} + \frac{1441952704}{14105329875}z^{12}$$
$$-\frac{893393408}{9499507875}z^{13} + \frac{9352282112}{107417512125}z^{14} - \cdots \tag{5.12}$$

is non-trivial solution of $(z+1)e^{-z} = (S+1)e^{-S}$ in the vicinity of ramification point. Note that in these notations $\omega_1(z')\omega_2(\tilde{z}') = \breve{\omega}_1(z')\breve{\omega}_2(S(z'))\frac{dS(z')}{dz'}(dz')^2$ where $\omega(z) = \breve{\omega}(z)dz$, i.e.

$$< \omega_1|\omega_2 >_z = dz \oint_{z'=0} \frac{\frac{1}{z-z'} - \frac{1}{z-\tilde{z}'}}{\log(1+z') - \log(1+\tilde{z}')} \frac{1+z'}{z'} \frac{dS(z')}{dz'} \breve{\omega}_1(z')\breve{\omega}_2(S(z'))dz'$$
$$= < \omega_2|\omega_1 >_z . \tag{5.13}$$

Our star product differs by a factor of 2 from that in [3].

The next suggestion of [3] is to take

$$\zeta_k(z) = \frac{zdz}{1+z}\left(-\frac{1+z}{z}\frac{d}{dz}\right)^{n+2}z \tag{5.14}$$

i.e.

$$\zeta_{-1} = dz, \quad \zeta_0 = \frac{dz}{z^2}, \quad \zeta_1 = \frac{2z+3}{z^4}dz, \quad \zeta_2 = \frac{6z^2+20z+15}{z^6}dz,$$
$$\zeta_3 = \frac{24z^3+130z^2+210z+105}{z^8}dz, \zeta_4 = \frac{120z^4+924z^3+2380z^2+2520z+945}{z^{10}}dz,$$
$$\zeta_5 = \frac{720z^5+7308z^4+26432z^3+44100z^2+34650z+10395}{z^{12}}dz,$$
$$\zeta_6 = \frac{5040z^6+64224z^5+303660z^4+705320z^3+866250z^2+540540z+135135}{z^{14}}dz,$$
$$\zeta_7 = \frac{40320z^7+623376z^6+3678840z^5+11098780z^4+18858840z^3+18288270z^2+9459450z+2027025}{z^{16}}dz,$$
$$\cdots$$
$$\zeta_k = (k+1)!\frac{dz}{z^{k+2}}\left(1+O(z^{-1})\right) \tag{5.15}$$

Accordingly, from (5.4) and (5.10),

$$v_0 = dz, \quad v_1 = zdz, \quad v_2 = \frac{z^2-z}{2}dz, \quad v_3 = \frac{2z^3-5z^2+5z}{12}dz, \quad v_4 = \frac{1}{120}(5z^4dz - 15v_2 - 130v_3),$$

$$\cdots$$

$$v_k = \frac{z^kdz}{k!}\left(1+O(z^{-1})\right) \tag{5.16}$$

It is now easy to evaluate the products of v and ζ differentials, the lowest products are listed in table 2. Consequently,

$$v_k * \zeta_{k-l} = 0 \quad \text{for} \quad l \geq 2,$$

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$$
\begin{aligned}
v_k * \zeta_{k-1} &= \zeta_0 \,, \\
v_k * \zeta_k &= \frac{2k+1}{3}\zeta_1 \,, \\
v_k * \zeta_{k+1} &= \frac{(2k+1)(2k+3)}{15}\zeta_2 - \frac{2(k-1)(k+3)}{5\cdot 27}\zeta_1 \,, \\
v_k * \zeta_{k+2} &= \frac{(2k+1)(2k+3)(2k+5)}{105}\zeta_3 - \frac{4(k-1)(k+4)(2k+3)}{27\cdot 35}\zeta_2 \\
&\quad + \frac{2(k-1)(k+4)(2k+3)}{3^5\cdot 35}\zeta_1 \,, \\
v_k * \zeta_{k+3} &= \frac{(2k+1)(2k+3)(2k+5)(2k+7)}{945}\zeta_4 + \cdots \,,
\end{aligned}
$$

$$
\cdots . \tag{5.17}
$$

Finally,

$$
\frac{1}{2}Tr_* B(\bullet,\bullet) = \frac{1}{24}(\zeta_1 - \zeta_0)\,. \tag{5.18}
$$

Note that it can not be represented simply as $\sum_{n=0}^{\infty}(v_n * \zeta_n)(z)$: the sum diverges, but contour integral provides a self-consistent expression for this quantity, which notably includes ζ_0, not only ζ_1 (!).

5.3 AMM-Eynard equations for Lambert curve

All this implies that the AMM-Eynard equation (5.1), (5.7) for the Lambert curve and the BM choice of $\{\zeta_n\}$ has the form

$$
\sum_{m=0}^{\infty} \frac{2^{m-1}}{(2m+1)!!} \zeta_m \hat{\mathcal{M}}_{m-1}^{\mathrm{BM}} \mathcal{Z}_{\mathrm{BM}} = 0 \tag{5.19}
$$

where

$$
\begin{aligned}
\hat{\mathcal{M}}_{-1}^{\mathrm{BM}} &= \sum_{k=1}^{\infty} \tilde{T}_k \frac{\partial}{\partial T_{k-1}} + \frac{1}{2}T_0^2 - \frac{1}{24} = L_{-1} - \frac{1}{24} \\
2\hat{\mathcal{M}}_0^{\mathrm{BM}} &= 2\left\{ \hat{L}_0 - \frac{1}{3}\left(\frac{16}{15}\hat{L}_1 - \hat{N}_1 \right) + \frac{2}{5\cdot 81}\left(\frac{8}{7}\hat{L}_2 - \hat{N}_2 \right) + \cdots \right\} \\
4\hat{\mathcal{M}}_1^{\mathrm{BM}} &= 4\hat{L}_1 - \frac{64}{9\cdot 7}\hat{L}_2 + \frac{8}{9}\hat{N}_2 = 4\left\{ \hat{L}_1 - \frac{2}{9}\left(\frac{8}{7}\hat{L}_2 - \hat{N}_2 \right) + \cdots \right\} \\
&\quad \cdots .
\end{aligned} \tag{5.20}
$$

The constant shift by $1/24$ in $\mathcal{M}_{-1}^{\mathrm{BM}}$ comes from the *anomalous* contribution with ζ_0 to $\frac{1}{2}Tr_* B(\cdot,\cdot)$. The ζ_1-term in the same trace contributes the usual $3/24 = 1/8$ to $2L_0$. The shifted times are $\tilde{T}_k = T_k - \delta_{k,1}$. The new operators at the r.h.s. are:

$$
\begin{aligned}
\hat{N}_1 &= \sum_{k=0}^{\infty}(k+1)^2 \tilde{T}_k \frac{\partial}{\partial T_{k+1}}, \\
\hat{N}_2 &= \sum_{k=0}^{\infty}(k+1)(k+3/2)(k+2)\tilde{T}_k \frac{\partial}{\partial T_{k+2}}, \\
&\quad \cdots .
\end{aligned} \tag{5.21}
$$

Higher $\hat{N}$-operators can will also contain terms with second derivatives.

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A*B	ζ_0	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5
z^5	0	0	$35\zeta_0$	$\frac{245}{3}\zeta_1 + 154\zeta_0$	$147\zeta_2 + \frac{2\cdot7\cdot93}{15}\zeta_1 + 120\underline{\zeta_0}$	$231\zeta_3 + \frac{2\cdot7\cdot1069}{15}\zeta_2 + \frac{4\cdot31\cdot107}{3^2\cdot5}\underline{\zeta_1}$
z^4	0	$3\zeta_0$	$5\zeta_1 + 26\zeta_0$	$7\zeta_2 + \frac{32\cdot17}{9}\zeta_1 + 24\underline{\zeta_0}$	$9\zeta_3 + \frac{2\cdot163}{3}\zeta2 + \frac{4\cdot35\cdot13}{27}\underline{\zeta_1}$	$11\zeta_4 + \frac{512}{3}\zeta_3 - + \frac{4\cdot1627}{45}\underline{\zeta2} - \frac{8\cdot337}{3^4\cdot5}\zeta_1$
z^3	0	$5\zeta_0$	$\frac{25}{3}\zeta_1 + 6\underline{\zeta_0}$	$\frac{35}{3}\zeta_2 + \frac{16\cdot23}{27}\underline{\zeta_1}$	$15\zeta_3 + \frac{2\cdot547}{45}\underline{\zeta_2} - \frac{4\cdot103}{5\cdot81}\zeta_1$	$\frac{55}{3}\zeta_4 + \frac{16\cdot107}{45}\underline{\zeta_3} - \frac{4\cdot103}{27\cdot5}\zeta_2 + \frac{8\cdot37}{3^5\cdot5}\zeta_1$
z^2	ζ_0	$\zeta_1 + 2\underline{\zeta_0}$	$\zeta_2 + \frac{10}{3}\underline{\zeta_1}$	$\zeta_3 + \frac{14}{3}\underline{\zeta_2} - \frac{4}{27}\zeta_1$	$\zeta_4 + 6\underline{\zeta_3} - \frac{16}{45}\zeta_2 + \frac{8}{5\cdot81}\zeta_1$	$\zeta_5 + \frac{22}{3}\underline{\zeta_4} - \frac{28}{45}\zeta_3 + \frac{8}{27\cdot5}\zeta_2 + \frac{32}{3^5\cdot5}\zeta_1$
z	$\underline{\zeta_0}$	$\underline{\zeta_1}$	$\underline{\zeta_2}$	$\underline{\zeta_3}$	$\underline{\zeta_4}$	$\underline{\zeta_5}$
1	$\frac{1}{3}\underline{\zeta_1}$	$\frac{1}{5}\underline{\zeta_2} + \frac{2}{45}\zeta_1$	$\frac{1}{7}\underline{\zeta_3} + \frac{16}{35\cdot9}\zeta_2 - \frac{8}{35\cdot81}\zeta_1$	$\frac{1}{9}\underline{\zeta_4} + \frac{10}{7\cdot27}\zeta_3 - \frac{4}{27\cdot35}\zeta_2 - \frac{8}{3^5\cdot7}\zeta_1$	$\frac{1}{11}\underline{\zeta_5} + \frac{16}{27\cdot11}\zeta_4 - \frac{4\cdot43}{81\cdot35\cdot11}\zeta_3 - \frac{8\cdot179}{81\cdot25\cdot77}\zeta_2 + \frac{16\cdot71}{3^6\cdot25\cdot77}\zeta_1$	$\frac{1}{13}\underline{\zeta_6} + \frac{70}{99\cdot13}\zeta_5 - \frac{8\cdot49}{81\cdot55\cdot13}\zeta_4 - \frac{8\cdot71\cdot157}{3^5\cdot25\cdot77\cdot13}\zeta_3 + \frac{16\cdot2447}{3^6\cdot25\cdot77\cdot13}\zeta_2 + \frac{32\cdot67}{3^8\cdot11\cdot13}\zeta_1$
ζ_0	$\frac{1}{15}\underline{\zeta_2} - \frac{8}{5\cdot27}\zeta_1$	$\frac{1}{35}\underline{\zeta_3} - \frac{8}{35\cdot9}\zeta_2 + \frac{4}{35\cdot81}\zeta_1$	$\frac{1}{7\cdot9}\underline{\zeta_4} - \frac{16}{27\cdot35}\zeta_3 + \frac{16}{35\cdot81}\zeta_1$	$\frac{1}{3^2\cdot11}\underline{\zeta_5} - \frac{5\cdot16}{3^4\cdot7\cdot11}\zeta_4 - \frac{4}{3^5\cdot5\cdot11}\zeta_3 + \frac{8\cdot17\cdot71}{3^6\cdot5^2\cdot7\cdot11}\zeta_2 - \frac{32\cdot179}{3^8\cdot5^2\cdot7\cdot11}\zeta_1$	$\frac{1}{11\cdot13}\underline{\zeta_6} - \frac{8\cdot5}{3^3\cdot11\cdot13}\zeta_5 - \frac{512}{3^5\cdot5\cdot7\cdot11\cdot13}\zeta_4 + \frac{2^3\cdot17\cdot167}{3^6\cdot5^2\cdot11\cdot13}\zeta_3 - \frac{2^4\cdot47}{3^7\cdot5\cdot7\cdot13}\zeta_2 - \frac{2^7\cdot10099}{3^9\cdot5^2\cdot7\cdot11\cdot13}\zeta_1$	not
ζ_1		$\frac{1}{105}\underline{\zeta_4} - \frac{2}{9\cdot35}\zeta_3 + \frac{4}{35\cdot25}\zeta_2 - \frac{16}{81\cdot35}\zeta_1$	$\frac{1}{3\cdot7\cdot11}\underline{\zeta_5} - \frac{32}{3^3\cdot5\cdot7\cdot11}\zeta_4 + \frac{8}{3^4\cdot7\cdot11}\zeta_3 - \frac{2^4\cdot67}{3^5\cdot5^2\cdot7\cdot11}\zeta_2 + \frac{2^4\cdot29}{3^7\cdot5^2\cdot7\cdot11}\zeta_1$	$\frac{1}{3\cdot11\cdot13}\underline{\zeta_6} - \frac{5^2\cdot2}{3^3\cdot7\cdot11\cdot13}\zeta_5 + \frac{2^3\cdot139}{3^5\cdot5\cdot7\cdot11\cdot13}\zeta_4 - \frac{2^4\cdot3^3\cdot5\cdot23}{3^6\cdot5^2\cdot7\cdot11\cdot13}\zeta_3 - \frac{2^4\cdot37}{3^5\cdot5^2\cdot7\cdot11\cdot13}\zeta_2 + \frac{2^7\cdot103}{3^7\cdot5^2\cdot7\cdot13}\zeta_1$	enough	space

Table 2: The lowest *-products of ξ_k's and z^k's.

Indeed, let us continue formulae (5.20) further. Then, picking up the terms with $\frac{1}{3}\zeta_1$ in (5.17) one obtains

$$2\hat{\mathcal{M}}_0^{\mathrm{BM}} = \underbrace{\frac{1}{8} + \sum_{k=0}^{\infty}(2k+1)\tilde{T}_k\frac{\partial}{\partial T_k}}_{2\hat{L}_0} - \frac{2}{45}\sum_{k=0}^{\infty}(k-1)(k+3)\tilde{T}_k\frac{\partial}{\partial T_{k+1}} - \frac{4}{45}\frac{\partial^2}{\partial T_0^2} + \cdots =$$

$$= 2\hat{L}_0 - \frac{8}{45}\underbrace{\left(\sum_{k=0}^{\infty}(2k+1)(2k+3)\tilde{T}_k\frac{\partial}{\partial T_{k+1}} + \frac{1}{2}\frac{\partial^2}{\partial T_0^2}\right)}_{4\hat{L}_1}$$

$$+ \frac{2}{45}\underbrace{\sum_{k=0}^{\infty}\Big(4(2k+1)(2k+3) - (k-1)(k+3)\Big)\tilde{T}_k\frac{\partial}{\partial T_{k+1}} + \cdots =}_{15\hat{N}_1}$$

$$= 2\hat{L}_0 - \frac{32}{45}\hat{L}_1 + \frac{2}{3}\hat{N}_1 + \cdots. \tag{5.22}$$

The operators appearing in $\mathcal{M}_n$'s are naturally graded with assigning weight $2k+1$ to T_k. Then, the next (grade 2) contribution to the same term $2\mathcal{M}_0^{\mathrm{BM}}$ is similarly

$$\frac{1}{35 \cdot 81}\left(\sum_k 2(k-1)(k+4)(2k+3)\tilde{T}_k\frac{\partial}{\partial T_{k+2}} + 12\frac{\partial^2}{\partial T_0\partial T_1}\right) =$$

$$= \frac{4}{35 \cdot 81}\underbrace{\left(\sum_k(2k+1)(2k+3)(2k+5)\tilde{T}_k\frac{\partial}{\partial T_{k+2}} + 3\frac{\partial^2}{\partial T_0\partial T_1}\right)}_{8\hat{L}_2}$$

$$- \frac{2}{5 \cdot 81}\underbrace{\sum_k(k+1)(k+2)(2k+3)\tilde{T}_k\frac{\partial}{\partial T_{k+2}}}_{2\hat{N}_2}. \tag{5.23}$$

Further, the grade 3 contribution to $\mathcal{M}_0^{\mathrm{BM}}$ (i.e. to the coefficient in front of $\frac{1}{3}\zeta_1$) will be

$$-\frac{2^3 \cdot 5}{35 \cdot 81}\tilde{T}_0\frac{\partial}{\partial T_3} + 0 \cdot \tilde{T}_1\frac{\partial}{\partial T_4} + \frac{2^4 \cdot 7}{35 \cdot 81}\tilde{T}_2\frac{\partial}{\partial T_5} + \frac{2^5 \cdot 11}{35 \cdot 81}\tilde{T}_3\frac{\partial}{\partial T_6}$$

$$+ \cdots + 2 \cdot 3 \cdot \frac{16}{35 \cdot 81}\frac{\partial^2}{\partial T_0\partial T_2} - 3 \cdot \frac{16}{35 \cdot 81}\frac{\partial^2}{\partial T_1^2} \tag{5.24}$$

while

$$16\hat{L}_3 = \sum_k(2k+1)(2k+3)(2k+5)(2k+7)\tilde{T}_k\frac{\partial}{\partial T_{k+3}} + 15\frac{\partial^2}{\partial T_0\partial T_2} + \frac{9}{2}\frac{\partial^2}{\partial T_1^2}. \tag{5.25}$$

If one arranges to eliminate the T_0T_{m-1} derivative from $\hat{N}_m$, then $\hat{N}_3$ will be a combination of $\hat{n}_3$ (which is defined not to contain the second derivatives in times)) and $\frac{\partial^2}{\partial T_1^2}$.

Collect now the terms with $\frac{1}{15}\zeta_2$:

$$4\hat{\mathcal{M}}_1^{\mathrm{BM}} = \underbrace{\sum_k (2k+1)(2k+3)\tilde{T}_k \frac{\partial}{\partial T_{k+1}} + \frac{1}{2}\frac{\partial^2}{\partial T_0^2}}_{4\hat{L}_1}$$

$$-\frac{4}{9\cdot 7}\left(\sum_k (k-1)(k+4)(2k+3)\tilde{T}_k \frac{\partial}{\partial T_{k+2}} + 6\frac{\partial^2}{\partial T_0 \partial T_1}\right) + \cdots =$$

$$= 4\hat{L}_1 - \frac{8}{9\cdot 7}\underbrace{\left(\sum_k (2k+1)(2k+3)(2k+5)\tilde{T}_k \frac{\partial}{\partial T_{k+2}} + 3\frac{\partial^2}{\partial T_0 \partial T_1}\right)}_{8\hat{L}_2}$$

$$+\frac{4}{9\cdot 7}\underbrace{\sum_k \Big(2(2k+1)(2k+3)(2k+5) - (k-1)(k+4)(2k+3)\Big)\tilde{T}_k \frac{\partial}{\partial T_{k+2}}}_{14\hat{N}_2} + \cdots$$

$$= 4\hat{L}_1 - \frac{64}{9\cdot 7}\hat{L}_2 + \frac{8}{9}\hat{N}_2 + \cdots . \tag{5.26}$$

5.4 New algebra

Thus, we are led to a new set of the operator $\hat{N}_1$ and its descendants $\hat{N}_2, \ldots$ produced by its commuting with the Virasoro algebra. Together with the Virasoro algebra, they form an extended algebra of operators $\hat{A}_m$. Denote the linear (in derivatives) part of these operators through $\hat{a}_m = \sum_k P_m(k)\tilde{T}_k \frac{\partial}{\partial T_{k+m}}$. The polynomials $P_m(k)$ are

$$
\begin{aligned}
\hat{L}_{-1} &: \quad 1 \\
\hat{L}_0 &: \quad \frac{1}{2}(2k+1) \\
\hat{L}_1 &: \quad \frac{1}{4}(2k+1)(2k+3) \\
&\quad \ldots \\
\hat{N}_1 &: \quad (k+1)^2 \\
\hat{N}_2 &: \quad 2(k+1)(k+\frac{3}{2})(k+2) \\
&\quad \ldots .
\end{aligned}
\tag{5.27}
$$

One has for the commutator $[\hat{a}_m, \hat{a}_n]$

$$P_{m+n}(k) = P_m(k)P_n(k+m) - P_n(k)P_m(k+n). \tag{5.28}$$

Using this rule, one obtains

$$
\begin{aligned}
\left[\hat{N}_1, \hat{L}_{-1}\right] &= 2\left(\hat{L}_0 - \frac{1}{16}\right), \\
\left[\hat{N}_1, \hat{L}_0\right] &= \hat{N}_1, \\
\left[\hat{N}_1, \hat{L}_1\right] &= \frac{2}{3}\left(\hat{N}_2 - \hat{L}_2\right), \quad \text{(quadratic pieces also match)}
\end{aligned}
$$

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$$\cdots \tag{5.29}$$

$$\left[\hat{N}_2, \hat{L}_{-1}\right] = 3\hat{N}_1 \,,$$

$$\left[\hat{N}_2, \hat{L}_0\right] = 2\hat{N}_2 \,.$$

The commutator

$$\left[\hat{N}_1, \hat{N}_2\right] = -2\sum_k (k+1)(k+2)^2(k+3)\tilde{T}_k \frac{\partial}{\partial T_{k+3}} = \hat{N}_3 \tag{5.30}$$

gives rises to a new operator $\hat{N}_3$ etc.

5.5 Introduction of u^2

Let us manifestly restore in the formulas the dependence on the deformation parameter u.

The terms of order u^0 are underlined in the table. Those to the right come with higher powers of u^2. Those to the left come with powers of $1/u^2$, but they are eliminated in linear combinations of z^k, needed to produce v_k.

5.6 Commutation relations

For

$$\hat{\mathcal{M}}_{-1} = \hat{L}_{-1} - \frac{1}{24}u^2 \,, \tag{5.31}$$

and

$$2\hat{\mathcal{M}}_0 = 2\hat{L}_0 + u^2(\alpha_{01}\hat{L}_1 - \nu_{01}\hat{N}_1) + u^4(\alpha_{02}\hat{L}_2 - \nu_{02}\hat{N}_2) + \cdots \tag{5.32}$$

one has

$$\left[\hat{\mathcal{M}}_0, 2\hat{\mathcal{M}}_{-1}\right] = 2\hat{L}_{-1} + u^2\Big(2\alpha_{01}\hat{L}_0 - \nu_{01}[\hat{N}_1, \hat{L}_{-1}]\Big) \tag{5.33}$$

$$+ u^4\Big(3\alpha_{02}\hat{L}_1 - \nu_{02}[\hat{N}_2, \hat{L}_{-1}]\Big) + \cdots \,.$$

Now the point is that, since

$$\left[\hat{N}_1, \hat{L}_{-1}\right] = \sum_{k=0}^{\infty}(2k+1)T_k\frac{\partial}{\partial T_k} = 2\hat{L}_0 - \frac{1}{8} \tag{5.34}$$

the u^2 term at the r.h.s. of (5.34) is actually $u^2\left(\frac{\nu_{01}}{8} + 2(\alpha_{01} - \nu_{01})\hat{L}_0\right)$, i.e. one may expect the r.h.s. of (5.34) is actually such that

$$2\left[\hat{\mathcal{M}}_0, \hat{\mathcal{M}}_{-1}\right] = 2\underbrace{\left(\hat{L}_{-1} - \frac{1}{24}u^2\right)}_{\hat{\mathcal{M}}_{-1}} + \text{const}\cdot u^2\hat{\mathcal{M}}_0 + \text{const}\cdot u^4\hat{\mathcal{M}}_1 + \cdots \tag{5.35}$$

and the operators $\hat{\mathcal{M}}_m$ form a closed algebra. This requires a conspiracy of the coefficients, say,

$$\frac{1}{8}\nu_{01} = -\frac{2}{24}, \quad \text{i.e.} \ \ \nu_{01} = -\frac{2}{3} \tag{5.36}$$

which is indeed the case.

5.7 From $\mathcal{M}_n$ to Virasoro algebra

An important property of this closed algebra of operators $\mathcal{M}_k$ is that it can be converted, with a triangular transformation, into the Virasoro algebra:

$$
\begin{aligned}
\hat{\mathcal{L}}_{-1} &= \hat{\mathcal{M}}_{-1} = \hat{L}_{-1} - \frac{u^2}{24}, \\
\hat{\mathcal{L}}_0 &= \hat{\mathcal{M}}_0 + \frac{2u^2}{45}\hat{\mathcal{M}}_1 + 0 \cdot u^4 \left(\frac{8}{7}\hat{L}_2 - \hat{N}_2\right) + \cdots = \hat{L}_0 - \frac{u^2}{3}(\hat{L}_1 - \hat{N}_1), \\
&\cdots
\end{aligned}
\tag{5.37}
$$

Then

$$
\left[\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_{-1}\right] = \hat{L}_{-1} - \frac{2u^2}{3}\hat{L}_0 + 2\left(\hat{L}_0 - \frac{1}{16}\right) + \cdots = \hat{L}_{-1} - \frac{u^2}{24} = \hat{\mathcal{L}}_{-1}.
\tag{5.38}
$$

Furthermore, it looks like

$$
\hat{\mathcal{L}}_m = \hat{U}\hat{L}_m\hat{U}^{-1}
\tag{5.39}
$$

where

$$
\hat{U} = \exp\left\{-\frac{u^2}{3}(\hat{L}_1 - \hat{N}_1) + O(u^6)\right\} = \exp\left\{\frac{u^2}{12}\left(\sum_{k=0}^{\infty}\tilde{T}_k\frac{\partial}{\partial T_{k+1}} - \frac{1}{2}\frac{\partial^2}{\partial T_0^2}\right) + O(u^6)\right\}
\tag{5.40}
$$

Indeed,

$$
\begin{aligned}
\hat{U}\hat{L}_{-1}\hat{U}^{-1} &= \hat{L}_{-1} - \frac{u^2}{3}[(\hat{L}_1 - \hat{N}_1), \hat{L}_{-1}] + \frac{u^4}{18}\Big[(\hat{L}_1 - \hat{N}_1), [(\hat{L}_1 - \hat{N}_1), \hat{L}_{-1}]\Big] + \cdots = \\
&= \hat{L}_{-1} - \frac{u^2}{3}\left(2\hat{L}_0 - 2\Big(\hat{L}_0 - \frac{1}{16}\Big)\right) + 0 = \hat{L}_{-1} - \frac{u^2}{24} = \hat{\mathcal{L}}_{-1}
\end{aligned}
\tag{5.41}
$$

— since the first commutator is a c-number, all multiple commutators automatically vanish. Similarly,

$$
\begin{aligned}
\hat{U}\hat{L}_0\hat{U}^{-1} &= \hat{L}_0 - \frac{u^2}{3}[(\hat{L}_1 - \hat{N}_1), \hat{L}_0] + \frac{u^4}{18}\Big[(\hat{L}_1 - \hat{N}_1), [(\hat{L}_1 - \hat{N}_1), \hat{L}_0]\Big] + \cdots = \\
&= \hat{L}_0 - \frac{u^2}{3}(\hat{L}_1 - \hat{N}_1) + 0 = \hat{L}_0 - \frac{u^2}{3}(\hat{L}_1 - \hat{N}_1) = \hat{\mathcal{L}}_0
\end{aligned}
\tag{5.42}
$$

— again the form of the first commutator implies the vanishing of all multiple commutators.

Then, since the Kontsevich partition function Z_K is annihilated by $\hat{L}_{m\geq-1}$, while its Kontsevich-Hurwitz deformation $\mathcal{Z}$ by $\mathcal{M}_m$, one has

$$
\mathcal{Z}(u) = \hat{U}Z_K.
\tag{5.43}
$$

This implies a series of relations. Indeed,

$$
\begin{aligned}
\mathcal{Z} &= \left(1 - \frac{u^2}{3}(\hat{L}_1 - \hat{N}_1) + \frac{u^4}{18}(\hat{L}_1 - \hat{N}_1)^2 + \cdots\right)Z_K = \\
&= Z_K + \frac{u^2}{3}\hat{N}_1 Z_K + \frac{u^4}{18}\left(\frac{2}{3}\hat{N}_2 + \hat{N}_1^2\right)Z_K + \cdots.
\end{aligned}
\tag{5.44}
$$

We used here the fact that $\hat{L}_{m \geq -1}$ annihilate Z_K. In other words, one should have

$$F_1 = \frac{1}{3}\hat{N}_1 F_0\,,$$

$$F_2 + \frac{1}{2}F_1^2 = \frac{1}{18}\left(\frac{2}{3}\hat{N}_2 F_0 + \hat{N}_1^2 F_0 + (\hat{N}_1 F_0)^2\right)\,, \quad \text{i.e.} \quad F_2 = \frac{1}{18}\left(\frac{2}{3}\hat{N}_2 F_0 + \hat{N}_1^2 F_0\right)$$

$$\dots \qquad\qquad (5.45)$$

In particular, all low-genus contributions should vanish. For instance, one can check that

$$\hat{N}_1 F_0^{(0)} = \hat{N}_2 F_0^{(0)} = 0 \qquad\qquad (5.46)$$

and

$$\left(\frac{2}{3}\hat{N}_2 + \hat{N}_1^2\right) F_0^{(1)} = 0 \qquad\qquad (5.47)$$

$$F_1^{(1)} = \frac{1}{3}\hat{N}_1 F_0^{(1)}\,. \qquad\qquad (5.48)$$

Since at the same time

$$F_1^{(1)} = -\frac{1}{24}\partial_0^2 F_0^{(0)} \qquad\qquad (5.49)$$

we obtain an identity relating different genera of the Kontsevich partition function:

$$\hat{N}_1 F_0^{(1)} = -\frac{1}{8}\partial_0^2 F_0^{(0)} \qquad\qquad (5.50)$$

which supplements the first one in (5.46).

The next similar relations are

$$\hat{N}_1 F_1^{(1)} = \frac{1}{12}\partial_{01}^2 F_0^{(0)} \qquad\qquad (5.51)$$

$$F_2^{(2)} = \frac{1}{18}\left(\frac{2}{3}\hat{N}_2 + \hat{N}_1^2\right) F_0^{(2)} \qquad\qquad (5.52)$$

etc.

Note that action of the operator $\hat{\mathcal{L}}_0 = \hat{L}_0 - \frac{u^2}{3}(\hat{L}_1 - \hat{N}_1)$ on $\mathcal{Z} = \hat{U}Z_K$ can be represented by action of the operator

$$\hat{L}_0 + u^2 \partial_{u^2} \qquad\qquad (5.53)$$

which automatically guarantees that

$$\left[\hat{L}_0 + u^2 \partial_{u^2}, \hat{L}_{-1} - \frac{u^2}{24}\right] = \hat{L}_{-1} - \frac{u^2}{24}\,. \qquad\qquad (5.54)$$

However, *such* representation does not continue to higher $\hat{L}_{m \geq 1}$.

5.8 Annihilators of $F_0^{(0)}$

Relation (5.46) implies that there is a whole family of operators annihilating $F_0^{(0)}$. First of all, for each $m \geq 0$ there is a linear (without second derivatives) operator $\hat{\mathcal{N}}_m$: $\hat{\mathcal{N}}_0 = \hat{L}_0$, $\hat{\mathcal{N}}_1 = \hat{N}_1$, $\hat{\mathcal{N}}_2 = \hat{N}_2$, $\hat{\mathcal{N}}_3 = [\hat{N}_2, \hat{N}_1] = 2\sum_k (k+1)(k+2)^2(k+3)\tilde{T}_k \partial_{k+3}$ and so on. They all begin from

$$\hat{\mathcal{N}}_m \sim T_0 \partial_m + (m+3)(T_1 - 1)\partial_{m+1} + \cdots . \tag{5.55}$$

To illustrate how the other potential linear annihilators disappear, let us consider the level 3: there are three other annihilators of $F_0^{(0)}$ at this level:

$$1 \cdot 8 \cdot [\hat{N}_1, \hat{L}_2] = -2\sum_k (2k+3)(2k+5)(k^2+4k+1)\tilde{T}_k \partial_{k+3} - 3\left(\partial_1^2 + 4\partial_{02}^2\right),$$

$$2 \cdot 4 \cdot [\hat{N}_2, \hat{L}_1] = 2\sum_k (2k+3)(2k+5)(k+2)^2\tilde{T}_k \partial_{k+3} - 6\partial_{02}^2 , \tag{5.56}$$

$$16\hat{L}_3 = \sum_k (2k+1)(2k+3)(2k+5)(2k+7)\tilde{T}_k \partial_{k+3} + \frac{9}{2}\partial_1^2 + 15\partial_{02}^2 .$$

Quadratic derivatives cancel in a certain linear combination of these three lines, at the same time the linear part contains a factor $(2k+3)(2k+5)$, which can not be made consistent with (5.55). This implies that this linear combination should vanish identically. Indeed,

$$-12[\hat{N}_1, \hat{L}_2] + 4[\hat{N}_2, \hat{L}_1] - 16\hat{L}_3 = 0 . \tag{5.57}$$

When operator contains second time-derivatives, it acts on $F_0^{(0)}$ non-linearly: $\partial^2 \rightarrow \partial F \partial F$. Adding such non-linear annihilators we obtain $1 + \text{entier}\left(\frac{m+1}{2}\right)$ annihilators of $F_0^{(0)}$ at each level m. For $m = 1$ these are $\hat{N}_1$ and $\hat{L}_1$, for $m = 2$: $\hat{N}_2$ and $\hat{L}_2$, for $m = 3$: $\hat{\mathcal{N}}_3 \sim [\hat{N}_1, \hat{N}_2]$, $[\hat{L}_1, \hat{N}_2]$ and $\hat{L}_3$ (the fourth potential candidate, $[\hat{L}_2, \hat{N}_1]$ is a linear combination of the last two, as we already know).

6. Conclusion

To conclude, we demonstrated, at the level of convincing evidence rather than a rigorous proof, that the Kontsevich-Hurwitz partition function is annihilated by the Virasoro generators (2.9), which differ from the continuous Virasoro constraints by a conjugation.

Therefore, the KH partition function $\mathcal{Z}$ is now known to possess the following properties:

A. It is a generating function for the Hodge integrals [14, 15]

$$I_q^{(p)}(k_1, \ldots, k_m) \sim \int_{\overline{\mathcal{M}}_{p,m}} \lambda_q \psi_{k_1} \ldots \psi_{k_m} . \tag{6.1}$$

B. A change of time-variables $T \rightarrow T(p)$

$$T_k = u^{2k+1} \sum_{n=1}^{\infty} \frac{n^{n+k}}{n!} u^{3n} p_n \tag{6.2}$$

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converts $\mathcal{Z}$ into the Hurwitz partition function:

$$\mathcal{Z}\Big(T(p)\Big) = e^{H(p)} = e^{u^3\hat{W}_0}e^{p_1}, \tag{6.3}$$

where

$$\hat{W}_0 = \sum_{m=0}^{\infty} p_m \hat{V}_m = \frac{1}{2} \sum_{i,j\geq 1} \left((i+j)p_i p_j \frac{\partial}{\partial p_{i+j}} + ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} \right) \tag{6.4}$$

and $\hat{V}_m$ are the "discrete Virasoro" operators ($p_k = kt_k$)

$$\hat{V}_m = \sum_{k=0}^{\infty} (k+m)p_k \frac{\partial}{\partial p_{k+m}} + \sum_{i+j=m} ij \frac{\partial^2}{\partial p_i \partial p_j}. \tag{6.5}$$

C. For any fixed u and g, the partition function $\mathcal{Z}$ is a KP τ-function.

D. Associated multidensities (the generating functions of certain subsets of coefficients in $\mathcal{F}$) satisfy the AMM-Eynard equations on the Lambert spectral curve $x = (z+1)e^{-z}$.

E. $\mathcal{Z}$ is obtained from $Z_0 = e^{F_0}$ by the explicit u^2-dependent transformation

$$\mathcal{Z}(u,g) = \hat{U}(u,g)Z_K(g) \tag{6.6}$$

where

$$\hat{U} = \exp\left\{ \frac{u^2}{3}\Big(\hat{N}_1 - \hat{L}_1\Big) + O(u^6) \right\} = \exp\left\{ \frac{u^2}{12}\left(\sum_{k=0}^{\infty} T_k \frac{\partial}{\partial T_{k+1}} - \frac{g^2}{2}\frac{\partial^2}{\partial T_0^2} \right) + O(u^6) \right\} \tag{6.7}$$

and, consequently, it satisfies the Virasoro constraints

$$\hat{\mathcal{L}}_{m\geq -1}\mathcal{Z} = 0 \tag{6.8}$$

with

$$\hat{\mathcal{L}}_m = \hat{U}\hat{L}_m\hat{U}^{-1} \tag{6.9}$$

where $\hat{L}_{m\geq -1}$ are the usual "continuous Virasoro" operators [12], annihilating the Kontsevich partition function Z_K. This can be considered as a deformation of the Virasoro sub-algebra induced by the constant shift of the lowest $\hat{L}_{-1}$:

$$\hat{\mathcal{L}}_{-1} = \hat{L}_{-1} - \frac{u^2}{24} \tag{6.10}$$

and generated by the new important operator

$$\hat{N}_1 = \sum_{k=0}^{\infty} (k+1)^2 \tilde{T}_k \frac{\partial}{\partial T_{k+1}} \tag{6.11}$$

which annihilates the genus-zero Kontsevich free energy,

$$\hat{N}_1 F_0^{(0)} = 0. \tag{6.12}$$

The property **A** is the original problem, addressed by E.Witten and M.Kontsevich (at $q = 0$). It enters [1, 2] through the celebrated ELSV formulas.

The property **B** refers to representation of $H(p)$) through the action of "cut-and-join" operator $\hat{W}_0$, which was found in [18]. The relevant change of variables $T(p)$ is described in [2], see also [14].

The property **C** for the original Kontsevich model (at $u^2 = 0$, when the τ-function actually belongs to a narrow KdV class) was proved in [10, 13, 11] and was later studied by numerous different methods. For arbitrary u^2 it was proved by M.Kazarian. This is a non-trivial generalization from the $u^2 = 0$ case, in particular, the number of time variables in KP τ-function is effectively doubled as compared to the KdV one, and appropriate time-variables for $u^2 \neq 0$ are actually different from T (they are called q-variables in [2]). In fact, as we explained in section 4.3.4 above, once **B** is known, **C** is a simple direct corollary of the old theory of equivalent hierarchies [23, 24].

The property **D** was conjectured by V.Bouchard and M.Marino in [3]. They conjectured that the constraints are indeed quadratic (and thus reduced to the Virasoro, but not to some W-algebra) and associated with the Lambert curve, they also introduced the basis of ζ-differentials.

The property **E** has been our main concern in this paper. Note that the very fact that the generic Hodge integrals are somehow expressed through the intersection numbers, i.e. that the Kontsevich-Hurwitz partition function should be expressed through the Kontsevich one, is well known since [37] (based on earlier results due to D.Mumford). Our goal was to make this relation as explicit as possible.

We demonstrated in section 3 that the twisting (2.10)–(2.12) of the Kontsevich partition function *a la* [5] immediately reproduces (3.12)–(3.16) and explained in sections 4 and 5 how this fact is related to the previous works [1−3]. We do not provide rigorous proofs in this paper and concentrate instead on decisive evidence in support of (6.6) and (6.9). The reason for this is that once these relations are accepted, they can be used as the *better definition* of the Kontsevich-Hurwitz partition function. As explained in the Introduction, the definition provided by such reformulation is more fundamental than the original ones, as a generating function either of the Hodge integrals or of the Hurwitz numbers. Therefore, the detailed proof of (2.9) starting from the old definitions is, in fact, a problem of a rather limited interest, concerning the properties of moduli spaces or ramified coverings more than the theory of integrability and partition functions. Most of properties of the Kontsevich-Hurwitz partition function, including **A-D**, should now be derived directly from (2.9). We explained that parts of the relevant statements are already available in the matrix-model literature, still a complete derivation of **A-D** provides a set of important open problems.

Of greatest interest is the search for one more property **F**: an integral representation of $\mathcal{Z}$ – an appropriate u-dependent deformation of the Kontsevich matrix integral

$$Z_K = S(\Lambda) \int dX \exp\left\{ -\frac{2g^2}{27}\text{tr}X^3 + \text{tr}\Lambda^2 X \right\}, \quad T_k = \frac{(2k+1)!!}{2^k}\tau_k = \frac{(2k+1)!!}{2^k k}\text{tr}\Lambda^{-2k-1}$$

for which the relation to $H(p)$ should arise as a character expansion *a la* [31] and Virasoro constraints (2.9) should be the Ward identities, following from the reparametrization of

integration variables *a la* [20].

Far more straightforward should be three other exercises.

First, one can investigate Virasoro constraints for the Hurwitz function $\exp\left(H(p)\right)$ directly in terms of the p-variables and relate them to our $\hat{\mathcal{L}}_m$ through a change of variables — in the spirit of [38].

Second, of certain interest are generalizations to multi-Hurwitz free energies, which enumerate coverings of the Riemann sphere with several non-simple critical points (some results are already available on the combinatorial side for the case of two non-simple points). This research direction should be related to the celebrated conjecture about the Mumford measure on the universal moduli space made in the last chapter of [35].

Third, one can find the $\hat{U}$-operator, associated with the family

$$ X = (1-Z)Z^f \quad \text{or} \quad \left(1 - \frac{1}{f}\right)^f x = (1+z)\left(1 - \frac{1+z}{f}\right)^f $$

of spectral curves, of which the Lambert curve $x = (1+z)e^{-z}$ is the $f = \infty$ limit. This family is important for applications [39] and the relevant AMM-Eynard equations are already suggested in [3]. It remains "only" to repeat the consideration of our section 5.

In fact, as explained in [5], one expects that the continuous Virasoro algebra is relevant in the vicinity of any quadratic ramification point on a spectral curve, only the twisting operator $\hat{U}$ should be appropriately adjusted, and it is natural to expect that the quadratic AMM-Eynard equations on an arbitrary spectral curve describe some set of Virasoro constraints. Thus, the same formalism should work in many more cases. We understand that the same attitude is expressed in [7] (see, for example, the discussion of Mirzakhani relations in terms of the Virasoro algebra on the Weyl-Petersson curve $y = \frac{1}{2\pi}\sin(2\pi\sqrt{x})$ in these wonderful papers). Somewhat unexpected to us is a mysteriously simple form of the twisting operator $\hat{U}$ in the case of the Lambert curve, it would be interesting to see if this property persists in other important examples.

Acknowledgments

A. Diagram technique

In this appendix we give a detailed account of the diagram technique mentioned in section 4.2. It gives another interpretation of various quantities emerged in the context of generating functions for the Hurwitz numbers.
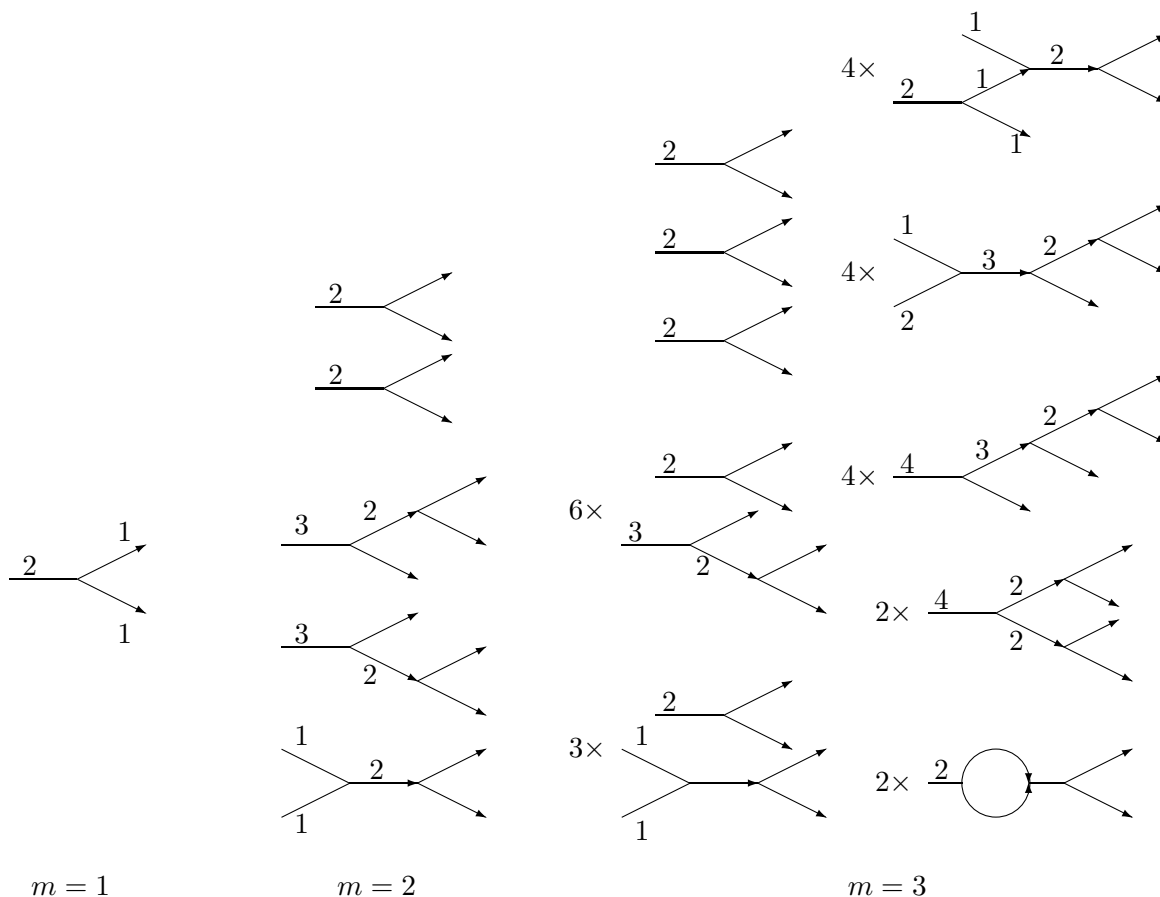
Figure 4: The lowest-order diagrams for Hurwitz function $H(p)$. All free arrows at the right hand are supposed to act on e^{p_1}, i.e. they carry index 1 (from p_1) and come with weight 1. Each diagram is a monomial in p_k's, where relevant values of k are indices of the incoming lines at the left. The sum of k's is equal to the number of free arrows in the diagram. Expression for diagram is made out of $ij/2$ and $(i+j)/2$ factors at the vertices and $\frac{u^{3m}}{m!}$ where m is the total number of vertices. Diagrams with p loops contribute only to $F^{(p)}$. Selection rule for q is more complicated, because u enters not only through u^{3m} but also through the $T(p)$ dependence. For $m = 3$ we do not draw identical diagrams, instead their multiplicity are shown.

A.0.1 Low-order terms in u

The lowest-order diagrams in figure 4 describe the first terms of the p-series $H(p)$:

$$\hat{W}_0 e^{p_1} = \frac{1}{2} p_2 e^{p_1},$$

$$\hat{W}_0^2 e^{p_1} = \frac{1}{4}\Big(p_2^2 + 2p_1^2 + 4p_3\Big)e^{p_1},$$

$$\hat{W}_0^3 e^{p_1} = \frac{1}{8}\Big(p_2^3 + 6p_1^2 p_2 + 12p_2 p_3 + 32p_1 p_2 + 32p_4 + 4p_2\Big)e^{p_1},$$

$$\hat{W}_0^4 e^{p_1} = \frac{1}{16}\Big(12p_1^4 + 12p_1^2 p_2^2 + 48p_1^2 p_3 + 64p_1^3 + 8p_1^2 + p_2^4 + 24p_3 p_2^2 +$$
$$+ 128p_2^2 p_1 + 208p_2^2 + 128p_4 p_2 + 48p_3^2 + 432p_3 p_1 + 144p_3 + 400p_5\Big)e^{p_1},$$
$$\dots . \tag{A.1}$$

Thus

$$H = p_1 + \frac{1}{2}u^3 p_2 + \frac{1}{4}u^6(2p_3 + p_1^2) + \frac{1}{12}u^9(8p_1p_2 + 8p_4 + p_2) + \cdots \tag{A.2}$$

$$H_{01} = p_1 + \frac{1}{2}u^3 p_2 + \frac{1}{2}u^6 p_3 + \frac{2}{3}u^9 p_4 + \cdots \tag{A.3}$$

$$H_{02} = \frac{1}{4}u^6 p_1^2 + \frac{2}{3}u^9 p_1 p_2 + u^{12}\left(\frac{1}{2}p_2^2 + \frac{9}{8}p_1 p_3\right) + u^{15}\left(\frac{32}{15}p_1 p_4 + \frac{9}{5}p_2 p_3\right) + \cdots \tag{A.4}$$

and

$$
\begin{aligned}
H - H_{01} - H_{02} ={}& \frac{1}{12}u^9 p_2 + u^{12}\left(\frac{3}{8}p_3 + \frac{1}{6}p_1^3 + \frac{1}{48}p_1^2\right) \\
&+ u^{15}\left(\frac{4}{3}p_4 + p_1^2 p_2 + \frac{1}{3}p_1 p_2 + \frac{1}{240}p_2\right) \\
&+ u^{18}\left(\frac{625}{144}p_5 + \frac{9}{4}p_1^2 p_3 + 2p_2^2 p_1 + \frac{27}{16}p_1 p_3 + \frac{2}{3}p_2^2 \right.\\
&\left. + \frac{1}{6}p_1^4 + \frac{9}{80}p_3 + \frac{1}{18}p_1^3 + \frac{1}{1440}p_1^2\right) + O(u^{21}).
\end{aligned}
\tag{A.5}
$$

A.0.2 Linear contributions to $\mathcal{F}$

Especially instructive is to compare the linear contributions to $H(p)$ and to $\mathcal{F}(T)$. The first T-linear terms in $\mathcal{F}$ are

$$
\begin{aligned}
\text{lin}\!\left(\mathcal{F}\right) ={}& \gamma_0^{(1)}T_1 + \gamma_0^{(2)}T_4 + \gamma_0^{(3)}T_7 + \cdots \\
&- \gamma_1^{(1)}T_0 - \gamma_1^{(2)}T_3 - \gamma_1^{(3)}T_6 - \cdots \\
&+ \gamma_2^{(2)}T_2 + \gamma_2^{(3)}T_5 + \cdots \\
&- \gamma_3^{(3)}T_4 - \cdots \\
&\cdots
\end{aligned}
\tag{A.6}
$$

and they all enter with different γ-factors, which are not fixed by the string equation (L_{-1}-constraint), only by the higher Virasoro constraints. Substituting $T(p)$ from (4.9), one obtains

$$
\begin{aligned}
\text{lin}\!\left(\mathcal{F}\right) &= \frac{1}{24}\sum_{n=1}^{\infty} u^{3+3n}p_n \frac{n^n}{n!}(n-1) + \sum_{n=1}^{\infty} u^{6+3n}p_n \frac{n^n}{n!}\left(\gamma_0^{(2)}n^4 - \gamma_1^{(2)}n^3 + \gamma_2^{(2)}n^2\right) + \cdots = \\
&= \sum_{p,n=1}^{\infty} u^{3p+3n}p_n \frac{n^n}{n!}\left(\gamma_0^{(p)}n^{3p-2} - \gamma_1^{(p)}n^{3p-3} + \cdots + (-)^p\gamma_p^{(p)}n^{2p-2}\right).
\end{aligned}
\tag{A.7}
$$

We explicitly substituted $\gamma_1^{(1)} = \gamma_0^{(1)} = \frac{1}{24}$ in the first term of the first relation in order to demonstrate that these two γ-factors are related so that p_1 drops away from $\mathcal{F}$. In fact this is true more generally: there are no p_1-linear terms in $\mathcal{F}\!\left(T(p)\right)$ at all, and this provides a set of relations for γ-factors (not a complete one, of course, moreover, the coefficients in front of T-linear terms do not exhaust the full set of γ-parameters).

Turning now to the Hurwitz free energy $H(p)$, its p-linear part is provided by "rooted" diagrams, with just one free leg at the left. Let us begin with the rooted trees. If r_m is the sum of all rooted tree diagrams with m vertices, then one has an obvious recurrent relation:

$$r_{m+1} = \frac{1}{2(m+1)} \sum_{i+j=m+2} is_{i-1} \cdot jr_{j-1} \tag{A.8}$$

i.e. $r(t) = \sum_{m=0}^{\infty} r_m t^m$ satisfies the differential equation

$$\partial_t r(t) = \frac{1}{2} \left\{ \partial_t \left(t \cdot r(t) \right) \right\}^2. \tag{A.9}$$

This gives $r = 1 + \frac{1}{2}t + \frac{1}{2}t^2 + \frac{2}{3}t^3 + \cdots = \sum_{n=1}^{\infty} \frac{n^{n-2}}{n!} t^{n-1}$ i.e. $\sum_{n=1}^{\infty} r_{n-1} p_n$ is exactly $H_{01}(p)$. Thus, we see that by subtracting H_{01} from $H(p)$ we throw away all the p-linear terms, coming from the tree diagrams. This fully eliminates the p_1-linear terms, because they can not come from loops, but other terms $p_{k \geq 2}$ can and do arise in $\mathcal{F}\left(T(p) \right)$.

Thus, one observes the appearance of peculiar series of the form $\sum_n \frac{n^{n+\alpha}}{n!} x^n$ (with $\alpha = -2$ in this particular case). Such series arise as inverse to Lambert-like functions. They are also important ingredient of the ELSV formula.

A.0.3 Developing diagram calculus

For future considerations (beyond the scope of the present paper) it is instructive to elaborate a little more on the diagram formalism. Summation of rooted trees is equivalent to evaluating $R(t)$,

$$e^{R(t)} = e^{t\hat{W}_0^-} e^{p_1} \tag{A.10}$$

where $t = u^3$ and $\hat{W}_0^- = \frac{1}{2} \sum_{i,j=1}^{\infty} ij p_{i+j} \partial_{ij}^2$ is a "half" of the $\hat{W}_0$ operator. Diagram analysis implies that $r(t)$ is a sum of *connected* diagrams, i.e. is linear in p-variables, so that $\partial_{ij}^2 r = 0$, moreover, conservation of "momentum" i at all vertices implies the selection rule

$$R(t) = \sum_{m=0}^{\infty} t^m r_m p_{m+1} = \oint x r(tx) dp(x) \tag{A.11}$$

$$\text{with} \quad r(x) = \sum_{m=0}^{\infty} r_m x^m \quad \text{and} \quad dp(x) = \sum_{k=1}^{\infty} \frac{p_k dx}{x^{k+1}}.$$

Therefore,

$$\dot{R} = e^{-R} \hat{W}_0^- e^R = \frac{1}{2} \sum_{i,j} ij p_{i+j} \partial_i R \partial_j R. \tag{A.12}$$

Substituting (A.12) and picking up the coefficient of t^m or, equivalently, of p_{m+1}, one reproduces eq. (A.8):

$$m r_m = \frac{1}{2} \sum_{i+j=m+1} ij r_{i-1} r_{j-1} \tag{A.13}$$

and (A.9):

$$\dot{r} = \frac{1}{2}\Big[(t \cdot r)^{\cdot}\Big]^2 . \tag{A.14}$$

After this reformulation one can easily do much more. For example, we can act with $\hat{W}_0^-$ not only on e^{p_1}, but, for example, on $e^{p_1 + \alpha_n p_n}$ and pick up the α-linear contribution. This would allow us to get an expression for the rooted tree, with exactly one of the outcoming arrows carrying index n (while all the rest still carry 1). The only thing to change in this case is the selection rule (A.12):

$$R(t, \alpha) = \sum_{m=0}^{\infty} t^m \Big(r_m p_{m+1} + \alpha_n r_m^{(n)} p_{m+n} + O(\alpha^2) \Big) \tag{A.15}$$

$$= \oint \Big(xr(tx) + \alpha_n x^n r'(tx)^{(n)} + O(\alpha^2) \Big) dp(x)$$

where we introduced the evident notation $r_m^{(n)}$ and keep to denote $r_m^{(1)}$ as r_m and so for $R^{(n)}(t)$ below. This immediately implies in addition to (A.13)

$$m r_m^{(n)} = \sum_{i+j=m+n} ij r_{i-1} r_{j-n}^{(n)} \tag{A.16}$$

— an already linear equation for $r(t)^{(n)}$ once $r(t)$ is known:

$$t^{n-1} \dot{r}^{(n)} = (t \cdot r^{(n)})^{\cdot} (t^n r^{(n)})^{\cdot} . \tag{A.17}$$

These equations are solved in terms of the peculiar special function $w(t)$, which belongs to the Lambert family and satisfies

$$t\dot{w} = w(1+w)^2 \tag{A.18}$$

and is the first member $w(t) = w_0(t)$ in the family of series

$$w_m(t) = \sum_{k=1}^{\infty} \frac{k^{k+m}}{k!} t^k . \tag{A.19}$$

From (A.18) it is easy to find expressions for all $w_m(t)$:

$$w_1 = t\dot{w}_0 = t\dot{w} = w(1+w)^2 ,$$
$$w_2 = t\dot{w}_1 = w(1+w)^2(1+4w+3w^2) ,$$
$$w_3 = t\dot{w}_2 = w(1+w)^4(1+10w+15w^2) , \tag{A.20}$$
$$\cdots$$
$$w_{m+1} = t\dot{w}_m = w(1+w)^2 \frac{dw_m}{dw} . \tag{A.21}$$

These functions will be used in section 4.3 below to define the transformations

$$T_m(p) = t^{\frac{2m+1}{3}} \oint w_m(tx) dp(x) .$$

One can also consider $w_m(t)$ with negative values of m. Since $t\dot{w}_{-1} = w_0 = w$, one obtains

$$w_{-1} = \frac{w}{1+w} \qquad (A.22)$$

because $\frac{d}{dw}\frac{w}{1+w} = \frac{w}{w(1+w)^2}$ and all the w series begin from t^1 (the absence of the t^0 term fixes the integration constants). This w_{-1} is the Lambert function *per se*. Similarly,

$$w_{-2} = \frac{w(2+w)}{2(1+w)^2} = w_{-1} - \frac{1}{2}w_{-1}^2 \qquad (A.23)$$

and so on.

We can now return to $r(t)$ and $r^{(n)}(t)$. Comparing the second formula for w_{-2} with eq. (A.14), we obtain:

$$r(t) = t^{-1}w_{-2}(t) = \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!}t^{k-1}, \quad \text{i.e.} \quad r_m = \frac{(m+1)^{m-1}}{(m+1)!}, \; m \geq 0. \qquad (A.24)$$

Indeed,

$$(t^{-1}w_{-2})^{\cdot} = t^{-2}(t\dot{w}_{-2} - w_{-2}) = t^{-2}(w_{-1} - w_{-2}) = \frac{1}{2t^2}w_{-1}^2 = \frac{1}{2}\dot{w}_{-2}^2$$

and this is exactly eq. (A.14) for $r = t^{-1}w_{-2}$.

Eq. (A.17) now acquires the form

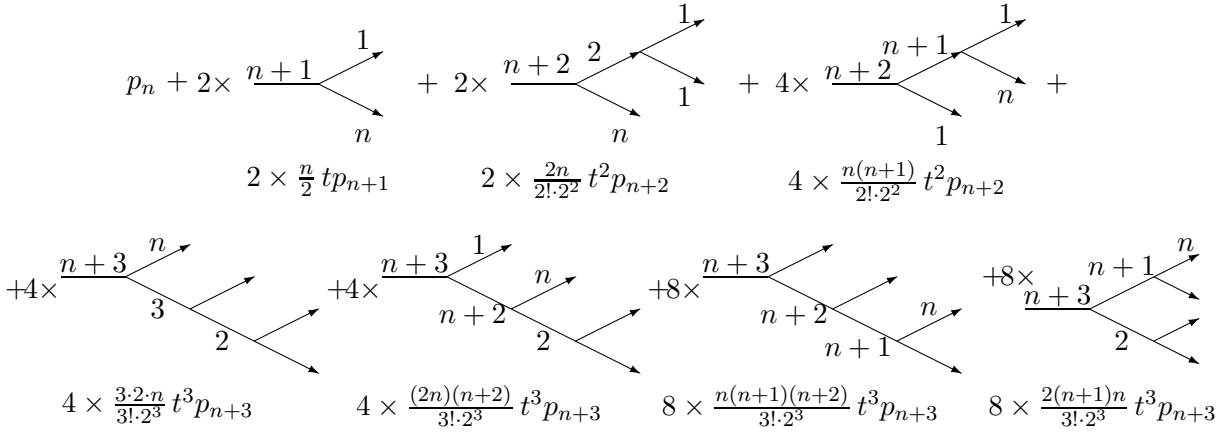$$t^n \dot{r}^{(n)} = w_{-1} \cdot (t^n r^{(n)})^{\cdot} \qquad \text{or} \qquad t \cdot \dot{r}^{(n)} = nr^{(n)}w \qquad (A.25)$$

which implies that

$$r^{(n)} = e^{nw_{-1}} = e^{\frac{nw}{1+w}} = 1 + nt + \frac{n(n+2)}{2}t^2 + \frac{n(n+3)^2}{6}t^3 + \cdots$$

$$= 1 + n\sum_{k=1}^{\infty} \frac{(n+k)^{k-1}}{k!}t^k,$$

$$R^{(n)} = \oint x^n r^{(n)}(tx)dp(x) = p_n + np_{n+1}t + \frac{n(n+2)}{2}p_{n+2}t^2 + \cdots$$

$$= p_n + n\sum_{k=1}^{\infty} \frac{(n+k)^{k-1}}{k!}t^k p_{n+k}. \qquad (A.26)$$

The first terms can be easily reproduced by direct evaluation of diagrams, see figure 5.

Similarly one can introduce and evaluate $R^{(m,n)}$, the sum of rooted trees with *two* outgoing arrows carrying indices m and n, and more generally,

$$R^{(n_1,\ldots,n_\nu)} = \sum_{k \geq 0} r_k^{(n_1,\ldots,n_\nu)} t^k p_{k+1+\sum_{i=1}^{\nu}(n_i-1)} = \oint x^{1+\sum_{i=1}^{\nu}(n_i-1)} r^{(n_1,\ldots,n_\nu)}(xt)dp(x),$$

$$r_k^{(n_1,\ldots,n_\nu)} = \frac{n_1 \ldots n_\nu m^{k-1}}{(k+1-\nu)!}, \quad m = k+1+\sum_{i=1}^{\nu}(n_i-1), \qquad (A.27)$$

$$\hat{R}^{(n_1,\ldots,n_\nu)} = \sum_{k,m=0}^{\infty} \frac{n_1 n_2 \ldots n_\nu m^{k-1}t^k}{(k+1-\nu)!}\delta\left(k+1+\sum_{i=1}^{\nu}(n_i-1) - m\right)p_m\partial_{n_1}\partial_{n_2}\ldots\partial_{n_\nu}$$

$$p_n + 2\times \quad\quad + 2\times \quad\quad + 4\times \quad\quad +$$

$$2 \times \tfrac{n}{2}\, tp_{n+1} \qquad 2 \times \tfrac{2n}{2!\cdot 2^2}\, t^2 p_{n+2} \qquad 4 \times \tfrac{n(n+1)}{2!\cdot 2^2}\, t^2 p_{n+2}$$

$$+4\times \qquad +4\times \qquad +8\times \qquad +8\times$$

$$4 \times \tfrac{3\cdot 2\cdot n}{3!\cdot 2^3}\, t^3 p_{n+3} \qquad 4 \times \tfrac{(2n)(n+2)}{3!\cdot 2^3}\, t^3 p_{n+3} \qquad 8 \times \tfrac{n(n+1)(n+2)}{3!\cdot 2^3}\, t^3 p_{n+3} \qquad 8 \times \tfrac{2(n+1)n}{3!\cdot 2^3}\, t^3 p_{n+3}$$

Figure 5: The simplest diagrams contributing to $r^{(n)}$: connected rooted trees with exactly one external arrow carrying index n. The coupling constant is $u^3 = t$. The sum of all diagrams is $R^{(n)} = p_n + np_{n+1}t + \frac{n(n+2)}{2}p_{n+2}t^2 + \frac{n(n+3)^2}{6}p_{n+3}t^3 + \cdots$ The zeroth-order contribution p_n corresponds to the diagram with no vertices, not shown on the picture. The combinatorial coefficient 8 in the last diagram is made from a "naive" factor 4, counting the places to attach the outgoing n arrow (or, what is the same, the "up-down" orientations of two vertices with different incoming lines) and an extra, perhaps less familiar 2, counting the "time"-ordering of the two most right vertices: a phenomenon illustrated also by appearance of B and C diagrams in figure 7 below.

where p_m is the momentum at the root, k — the number of vertices and the sum actually runs from $k = \nu - 1$. Of course, $R(t)$ and $R^{(n)}(t)$ are particular cases of this formula for $\nu = 0$ and $\nu = 1$ respectively. The corresponding generating functions are

$$\nu = 0 : r(t) = \sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{(k+1)!}\, t^k = t^{-1}w_{-2}\,,$$

$$\nu = 1 : r^{(n)}(t) = 1 + n\sum_{k=1}^{\infty} \frac{(n+k)^{k-1}}{k!}\, t^k = e^{nw_{-1}}\,,$$

$$\nu = 2 : r^{(n_1,n_2)}(t) = n_1 n_2 \sum_{k=1}^{\infty} \frac{(n_1+n_2+k-1)^{k-1}}{(k-1)!}\, t^k = t \cdot n_1 n_2 (1+w) e^{(n_1+n_2)w_{-1}}\,,$$

$$\nu = 3 : r^{(n_1,n_2,n_3)}(t) = n_1 n_2 n_3 \sum_{k=2}^{\infty} \frac{(n_1+n_2+n_3+k-2)^{k-1}}{(k-2)!}\, t^k$$
$$= t^2 \cdot n_1 n_2 n_3 (1+w)^2 (N+w) e^{Nw_{-1}}\,,$$

$$\nu = 4 : r^{(n_1,n_2,n_3,n_4)}(t) = n_1 n_2 n_3 n_4 \sum_{k=3}^{\infty} \frac{(n_1+n_2+n_3+n_4+k-3)^{k-1}}{(k-3)!}\, t^k =$$
$$= t^3 \cdot n_1 n_2 n_3 n_4 (1+w)^3 \Big(N^2 + (3N+1)w + 3w^2\Big) e^{Nw_{-1}}\,,$$

$$\nu = 5 : r^{(n_1,n_2,n_3,n_4,n_5)}(t) = n_1 n_2 n_3 n_4 n_5 \sum_{k=4}^{\infty} \frac{(N+k-4)^{k-1}}{(k-4)!}\, t^k =$$
$$= t^4 \cdot n_1 n_2 n_3 n_4 n_5 (1+w)^4 \Big(N^3 + (6N^2+4N+1)w + (15N+10)w^2 + 15w^3\Big) e^{Nw_{-1}}\,,$$

$$\nu = 6 : r^{(n_1, n_2, n_3, n_4, n_5, n_6)}(t) = n_1 n_2 n_3 n_4 n_5 n_6 \sum_{k=5}^{\infty} \frac{(N + k - 5)^{k-1}}{(k - 5)!} t^k =$$

$$= t^5 \cdot n_1 n_2 n_3 n_4 n_5 n_6 (1 + w)^5 \Big(N^4 + (10N^3 + 10N^2 + 5N + 1)w$$

$$+ (45N^2 + 60N + 25)w^2 + (105N + 105)w^3 + 105w^4 \Big) e^{Nw - 1} ,$$

$$\cdots \tag{A.28}$$

with $N = n_1 + n_2 + \cdots + n_\nu$. These "$(\nu + 1)$-point functions" are the main building blocks in the diagram technique. Knowing them, one can proceed to more complicated problems, for example, to double-root tree diagrams. They arise when one vertex of another kind, coming from the operator $\hat{W}_0^+ = \frac{1}{2} \sum_{i,j} (i + j) p_i p_j \partial_{i+j}$, is allowed. Thus, what we need is a linear-in-$\hat{W}_0^+$ term in

$$e^{t(\hat{W}_0^- + \hat{W}_0^+)} e^{p_1} = \left(e^{t\hat{W}_0^-} + \int_0^1 e^{st\hat{W}_0^-} t\hat{W}_0^+ e^{(1-s)t\hat{W}_0^-} ds + \cdots \right) e^{p_1} = \tag{A.29}$$

$$= \left(R(t) + \frac{t}{2} \sum_{i,j} (i + j) \int_0^1 ds \, e^{st\hat{W}_0^-} p_i p_j \frac{\partial}{\partial p_{i+j}} R((1 - s)t) + \cdots \right) e^{p_1}$$

where the Campbell-Hausdorf formula was used for the exponential:

$$e^{A+B} = e^A + \int_0^1 e^{sA} B e^{(1-s)A} ds + \int_0^1 ds_1 \int_0^{1-s_1} ds_2 e^{s_1 A} B e^{s_2 A} B e^{1-s_1-s_2} + \cdots .$$

In order to evaluate the action of the remaining operator we need $R^{(n)}$. The generating function for the connected double-rooted trees is

$$\frac{1}{2} \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\nu} \sum_{\substack{i,j=1 \\ l_1,\ldots,l_\nu=2}}^{\infty} (i+j) r_{i+j-1} r_{l_1-1} \ldots r_{l_\nu-1}$$

$$\times \int_0^1 [(1-s)t]^{i+j-1+\sum_{i=1}^{\nu}(l_i-1)} R^{(i,l_1,\ldots,l_\mu)}(ts) R^{(j,l_{\mu+1},\ldots,l_\nu)}(ts) t ds \tag{A.30}$$

see figure 6, where $\nu = 2$ additional trees are explicitly shown. It is straightforward to check that this expression reproduces H_{02}. In particular, figure 7 explicitly shows six diagrams, contributing to the $p_1 p_3$ double-rooted tree. Note that they are all different, for example, B and C differ by the "time" ordering of vertices, i.e. by the ordering of operators: $(\hat{W}_0^-)^2 \hat{W}_0^+ \hat{W}_0^+$ and $\hat{W}_0^- \hat{W}_0^+ (\hat{W}_0^-)^2$ respectively, moreover these two topologically equivalent diagrams enter with different combinatorial factors! This explicit example can serve to illustrate the peculiarities of Heitler diagram technique as compared to the more familiar Feynman one.

Similarly one can express through $R^{(m,n)}(t)$ the sum of rooted 1-loop diagrams and so on. This is a straightforward, though somewhat tedious procedure.
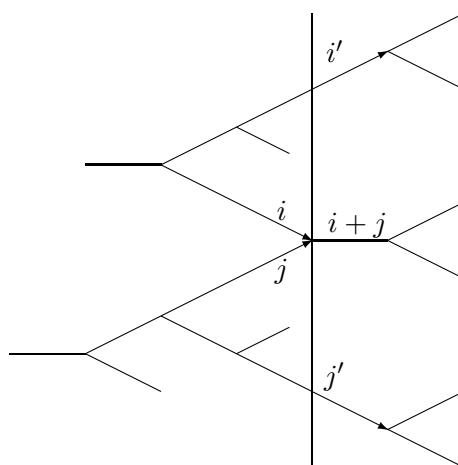
Figure 6: Schematic representation of a double-rooted graph contributing to (A.30). The vertical line shows the "time moment" of the action of operator $\hat{W}^+$. All the vertices to the right of this line ("before") come with the factor $(1-s)$, all the vertices to the left ("after") come with the factor s. The right parts of such diagrams are described in (A.30) by r-factors, the left parts by R's. Explicitly shown is the case of $\nu + 1 = 3$ disconnected branches evolved to the moment of $\hat{W}^+$ action, in general the sum over all ν should be made.
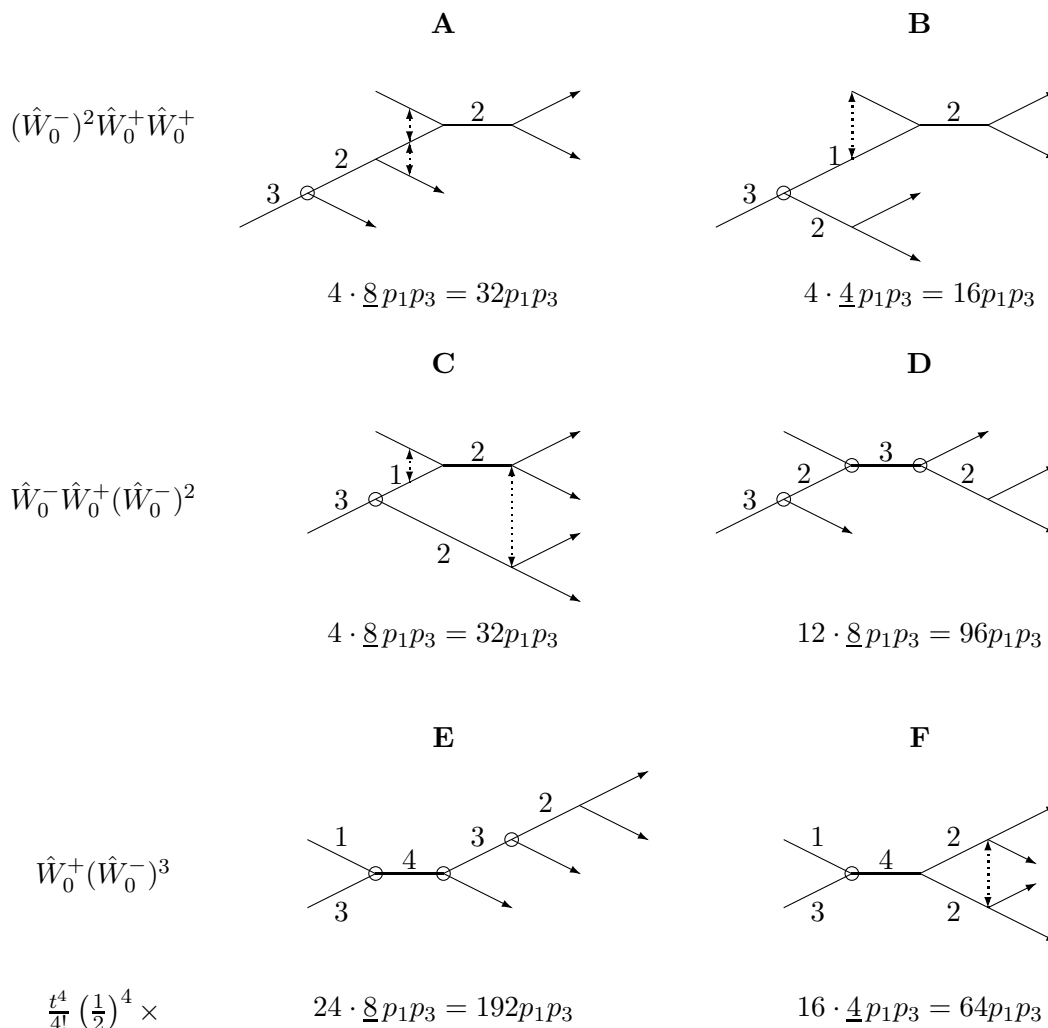
Figure 7: Six diagrams, contributing to the double-rooted tree p_1p_3 in eq. (A.30). Under each diagram its total contribution is written, — to be further multiplied by $\frac{1}{2^4 \cdot 4!}$ — and the coefficient in front of p_1p_3 is split into the "obvious" part, given by a product of ij and $i+j$ factors in the vertices and into "combinatorial factor" (underlined), counting the multiplicity of the diagram. When evaluating combinatorial factors one should also remember that vertices with $i < j$ should be counted twice, since they enter twice in the sum over i, j in the definition of $\hat{W}_0$. Such vertices are marked by a circle in the picture. Another contribution to combinatorial factor come from different possibilities to form the same diagram due to permutations of connections between vertices. Such permutations are shown by dotted lines. Each dotted line and each circle contribute a 2 to the combinatorial factor. Note that, since the two most right vertices in **F** are identical, no "time-ordering" factor of 2 contributes in this case. The sum of all the six diagrams is $\frac{16(2+1+2+6+12+4)}{2^4 \cdot 4!}p_1p_3 = \frac{27}{24}p_1p_3 = \frac{9}{8}p_1p_3$ and exactly reproduces the corresponding term in H_{02}.

References

[1] M. Kazarian and S.Lando, *An algebro-geometric proof of Witten's conjecture*, `math.AG/0601760`.

[2] M. Kazarian, *KP hierarchy for Hodge integrals*, talk at the *Workshop on Combinatorics of moduli spaces, Hurwitz numbers and cluster algebras*, Moscow, June 2–7 2008, http://www.mi.ras.ru/∼kazarian/papers/newwit0703.pdf.

[3] V. Bouchard and M. Mariño, *Hurwitz numbers, matrix models and enumerative geometry*, `arXiv:0709.1458`.

[4] A.S. Alexandrov, A. Mironov and A. Morozov, *Partition functions of matrix models as the first special functions of string theory. I: finite size Hermitean 1- matrix model*, *Int. J. Mod. Phys.* **A 19** (2004) 4127 [*Theor. Math. Phys.* **142** (2005) 349] [`hep-th/0310113`]; *Unified description of correlators in non-Gaussian phases of Hermitean matrix model*, *Int. J. Mod. Phys.* **A 21** (2006) 2481 [`hep-th/0412099`]; *Solving Virasoro constraints in matrix models*, *Fortschr. Phys.* **53** (2005) 512 [`hep-th/0412205`];
A.Alexandrov, A.Mironov, A.Morozov and P.Putrov, *Partition functions of matrix models as the first special functions of string theory. II. Kontsevich model*, to appear.

[5] A.S. Alexandrov, A. Mironov and A. Morozov, *M-theory of matrix models*, `hep-th/0605171`; *Instantons and merons in matrix models*, *Physica* **D 235** (2007) 126 [`hep-th/0608228`].

[6] B. Eynard, *Topological expansion for the 1-hermitian matrix model correlation functions*, *JHEP* **11** (2004) 031 [`hep-th/0407261`];
L. Chekhov and B. Eynard, *Hermitean matrix model free energy: feynman graph technique for all genera*, *JHEP* **03** (2006) 014 [`hep-th/0504116`].

[7] B. Eynard, *Universal distribution of random matrix eigenvalues near the "birth of a cut" transition*, *J. Stat. Mech.* (2006) P07005 [`math-ph/0605064`]; *Recursion between Mumford volumes of moduli spaces*, `arXiv:0706.4403`;
L. Chekhov, B. Eynard and N. Orantin, *Free energy topological expansion for the 2-matrix model*, *JHEP* **12** (2006) 053 [`math-ph/0603003`];
B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, `math-ph/0702045`; *Weil-Petersson volume of moduli spaces, Mirzakhani's recursion and matrix models*, `arXiv:0705.3600`;
N. Orantin, *From matrix models' topological expansion to topological string theories: counting surfaces with algebraic geometry*, PhD thesis, `arXiv:0709.2992`.

[8] A. Morozov, *String Theory: What is it?*, *Sov. Phys. Usp.* **35** (1992) 671 [*Usp. Fiz. Nauk.* **162** (1992) 83].

[9] A. Morozov, *Integrability and matrix models*, *Phys. Usp.* **37** (1994) 1 [`hep-th/9303139`]; *Challenges of matrix models*, `hep-th/0502010`;
A. Mironov, $2-D$ *gravity and matrix models. 1.* $2-D$ *gravity*, *Int. J. Mod. Phys.* **A 9** (1994) 4355 [`hep-th/9312212`]; *Matrix models of two-dimensional gravity*, *Phys. Part. Nucl.* **33** (2002) 537 [*Fiz. Elem. Chast. Atom. Yadra* **33** (2002) 1051].

[10] M.Kontsevich, *Theory of intersections on moduli space of curves*, *Funk. Anal. Prilozh.* **25** (1991) 50; *Intersection theory on the moduli space of curves and the matrix Airy function*, *Commun. Math. Phys.* **147** (1992) 1.

[11] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and A. Zabrodin, *Unification of all string models with C < 1*, *Phys. Lett.* **B 275** (1992) 311 [hep-th/9111037]; *Towards unified theory of 2-D gravity*, *Nucl. Phys.* **B 380** (1992) 181 [hep-th/9201013].

[12] M. Fukuma, H. Kawai and R. Nakayama, *Continuum Schwinger-Dyson equations and universal structures in two-dimensional quantum gravity*, *Int. J. Mod. Phys.* **A 6** (1991) 1385; R. Dijkgraaf, E. Verlinde and H. Verlinde, *Loop equations and Virasoro constraints in non perturbative 2d quantum gravity*, *Nucl. Phys.* **B 348** (1991) 435.

[13] A. Marshakov, A. Mironov and A. Morozov, *On equivalence of topological and quantum $2-D$ gravity*, *Phys. Lett.* **B 274** (1992) 280 [hep-th/9201011].

[14] I. Goulden, D. Jackson and R.Vakil, *The Gromov-Witten potential of a point, Hurwitz numbers and Hodge integrals*, *Proc. London Math. Soc.* **83** (2001) 563 [math.AG/9910004]; *A short proof of the λ_g-conjecture without Gromov-Witten theory: Hurwitz theory and the moduli of curves*, math.AG/0604297.

[15] B. Fantechi and R. Pandharipande, *Stable maps and branch divisors*, *Composition Math.* **130** (2002) 345 [math.AG/9905104];
M. Mirzakhani, *Weil-Petersson volumes and intersection theory on the moduli space of curves*, *J. Am. Math. Soc.* **20** (2007) 1;
Y. Kim and K. Liu, *A simple proof of Witten conjecture through localization*, math.AG/0508384;
M. Mulase and B. Safnuk, *Mirzakhani's recursion relations, Virasoro constraints and the KdV hierarchy*, math.QA/0601194.

[16] T. Ekedahl, S. Lando, M. Shapiro and A. Vainshtein, *On Hurwitz numbers and Hodge integrals*, *C. R. Acad. Sci. Paris Ser. I Math* **328** (1999) 1175 [math.AG/9902104]; *Hurwitz numbers and intersections on moduli spaces of curves*, *Invent. Math.* **146** (2001) 297 [math.AG/0004096];
S. Shadrin, *Geometry of meromorphic functions and intersections on moduli spaces of curves*, *Int. Math. Res. Notes* **38** (2003) 2051 [math.AG/0209282];
D. Zvonkine, *Enumeration of ramified coverings of the sphere and 2-dimensional gravity*, math.AG/0506248;
S. Shadrin and D. Zvonkine, *Changes of variables in ELSV-type formulas*, math.AG/0602457.

[17] A. Hurwitz, *Uber Riemann'sche Flachen mit gegebenen Verzweigungpunkten*, *Math. Ann.* **39** (1891) 1; *Uber die Anzal der Riemann'sche Flachen mit gegebenen Verzweigungpunkten*, *Math. Ann.* **55** (1902) 51;
S. Lando and D. Zvonkine, *On multiplicitites of the Lyashko-Looijenga mapping on the discriminant strata*, *Funk. Anal. Appl.* **33** (1999) 178; *Counting ramified coverings and intersection theory on spaces of rational functions. I*, math.AG/0303218;
S. Natanzon and V. Turaev, *A compactification of Hurwitz space*, *Topology* **38** (1999) 889;
S. Lando, *Ramified coverings of the two-dimensional sphere and intersection theory in spaces of meromorphic functions on algebraic curves*, *Russ. Math. Surv.* **57** (2002) 463;
A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz theory, and completed cycles*, *Ann. Math.* **163** (2006) 517 [math.AG/0204305];
T. Graber and R. Vakil, *Hodge integrals and Hurwitz numbers via virtual localization*, *Compositio Math.* **135** (2003) 25 [math.AG/0003028];
M. Kazarian and S. Lando, *Towards the intersection theory of Hurwitz spaces*, math.AG/0410388.

[18] I. Goulden and D. Jackson, *Transitive factorization into transpositions and holomorphic mappings on the sphere*, *Proc. Amer. Math. Soc.* **125** (1997) 51.

[19] R. Pandharipande, *The Toda equations and the Gromov-Witten theory of the Riemann sphere*, *Lett. Math. Phys.* **53** (2000) 59 [`math.AG/9912166`];
A.Okounkov, *Toda equations for Hurwitz numbers*, *Math. Res. Lett.* **7** (2000) 447 [`math.AG/0004128`];
A. Okounkov and R. Pandharipande, *Gromov-Witten the ory, Hurwitz numbers and matrix models. I*, `math.AG/0101147`.

[20] A. Mironov and A. Morozov, *On the origin of Virasoro constraints in matrix models: lagrangian approach*, *Phys. Lett.* **B 252** (1990) 47;
F. David, *Loop equations and nonperturbative effects in two- dimensional quantum gravity*, *Mod. Phys. Lett.* **A 5** (1990) 1019;
J. Ambjørn and Y.M. Makeenko, *Properties of loop equations for the hermitean matrix model and for two-dimensional quantum gravity*, *Mod. Phys. Lett.* **A 5** (1990) 1753;
H. Itoyama and Y. Matsuo, *Noncritical Virasoro algebra of d<1 matrix model and quantized string field*, *Phys. Lett.* **B 255** (1991) 202.

[21] A. Marshakov, A. Mironov and A. Morozov, *From Virasoro constraints in Kontsevich's model to W constraints in two matrix model*, *Mod. Phys. Lett.* **A 7** (1992) 1345 [`hep-th/9201010`];
Generalized matrix models as conformal field theories: discrete case, *Phys. Lett.* **B 265** (1991) 99;
A. Mironov and S. Pakulyak, *On the continuum limit of the conformal matrix models*, *Theor. Math. Phys.* **95** (1993) 604 [*Teor. Mat. Fiz.* **95** (1993) 317] [*Int. J. Mod. Phys.* **A 8** (1993) 3107] [`hep-th/9209100`];
S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and S. Pakuliak, *Conformal matrix models as an alternative to conventional multimatrix models*, *Nucl. Phys.* **B 404** (1993) 717 [`hep-th/9208044`].

[22] T. Shiota, *Characterization of Jacobian varieties in terms of soliton equations*, *Invent. Math.* **83** (1986) 333.

[23] S. Kharchev, A. Marshakov, A. Mironov and A. Morozov, *Landau-Ginzburg topological theories in the framework of GKM and equivalent hierarchies*, *Mod. Phys. Lett.* **A 8** (1993) 1047 [*Theor. Math. Phys.* **95** (1993) 571] [*Teor. Mat. Fiz.* **95** (1993) 280] [`hep-th/9208046`].

[24] S. Kharchev, *Kadomtsev-Petviashvili hierarchy and generalized Kontsevich model*, `hep-th/9810091`.

[25] A. Givental, *Gromov-Witten invariants and quantization of quadratic hamiltonians*, `math.AG/0108100`.

[26] M. Kazarian, *Remark on Hodge integrals and integrable hierarchies*, unpublished.

[27] D. Mumford, *Towards enumerative geometry on the moduli space of curves*, in *Arithmetics and Geometry. Volume II* M. Artin and J. Tate eds., Birkhäuser (1983) pg. 271–328.

[28] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, *Invent. Math.* **139** (2000) 173 [`math.AG/9810173`].

[29] M. Sato, *Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds*, *RIMS Kokyuroku* **439** (1981) 30;

E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *Transformation groups for soliton equations*, RIMS Symp., *Non-linear integrable systems — classical theory and quantum theory*, World scientific, Singapore (1983);

G. Segal and G. Wilson, *Loop groups and equations of KdV type*, *Publ. IHES.* **61** (1985) 5;

A. Morozov, *String theory and the structure of universal module space*, *JETP Lett.* **45** (1987) 585 [*Zh. Eksp. Teor. Fiz.* **45** (1987) 457] [*Phys. Lett.* **B 196** (1987) 325];

Y. Ohta, J. Satsuma, D. Takahashi and T. Tokihiro, *An Elementry Introduction to Sato Theory*, *Prog. Theor. Phys. Suppl.* **94** (1988) 210;

M. Fukuma, H. Kawai and R. Nakayama, *Infinite dimensional Grassmannian structure of two-dimensional quantum gravity*, *Commun. Math. Phys.* **143** (1992) 371.

[30] A. Morozov, *On the concept of universal W_∞ algebra*, *Sov. J. Nucl. Phys.* **51** (1990) 758 [*Nucl. Phys.* **B 357** (1991) 619] [*Yad. Fiz.* **51** (1990) 1190].

[31] S. Kharchev, A. Marshakov, A. Mironov and A. Morozov, *Generalized Kazakov-Migdal-Kontsevich model: group theory aspects*, *Int. J. Mod. Phys.* **A 10** (1995) 2015 [`hep-th/9312210`];

A. Mironov, A. Morozov and G.W. Semenoff, *Unitary matrix integrals in the framework of generalized Kontsevich model. 1. Brezin-Gross-Witten model*, *Int. J. Mod. Phys.* **A 11** (1996) 5031 [`hep-th/9404005`].

[32] Harish-Chandra, *Spherical functions on a semisimple Lie group I*, *Amer. J. Math.* **80** (1958) 241;

M. Olshanetsky and A. Perelomov, *Quantum Integrable Systems Related to Lie Algebras*, *Phys. Rept.* **94** (1983) 313;

F. Berezin and F. Karpelevich, *Zonal Spherical Functions and Laplace operators on some symmetric spaces*, *Dokl. Akad. Nauk SSSR* **118** (1958) 9;

D.J. Gross and W. Taylor, *Two-dimensional QCD is a string theory*, *Nucl. Phys.* **B 400** (1993) 181 [`hep-th/9301068`];

D.J. Gross and W. Taylor, *Twists and Wilson loops in the string theory of two- dimensional QCD*, *Nucl. Phys.* **B 403** (1993) 395 [`hep-th/9303046`];

V. Kazakov, M. Staudacher and T. Wynter, *Character expansion methods for matrix models of dually weighted graphs*, *Commun. Math. Phys.* **177** (1996) 451 [`hep-th/9502132`];

A. Okounkov, *Generating functions for intersection numbers on moduli spaces of curves*, `math.AG/0101201`;

A. Marshakov and N. Nekrasov, *Extended Seiberg-Witten theory and integrable hierarchy*, *JHEP* **01** (2007) 104 [`hep-th/0612019`].

[33] Yu. Chernyakov and V. Dolotin, *Operations and Identities in Tensor Algebra*, `math.AG/0501206`;

V. Dolotin and A. Morozov, *Introduction to non-linear algebra*, `hep-th/0609022`.

[34] E. Witten, *Two-dimensional gravity and intersection theory on moduli spaces*, *Surv. Diff. Geom.* **1** (1991) 243.

[35] V. Knizhnik, *Multiloop amplitudes in the theory of quantum strings and complex geometry*, *Sov. Phys. Usp.* **32** (1989) 945 [*Usp. Fiz. Nauk.* **159** (1989) 401].

[36] L. Chekhov, A. Marshakov, A. Mironov and D. Vasiliev, *DV and WDVV*, *Phys. Lett.* **B 562** (2003) 323 [`hep-th/0301071`];

L. Chekhov, A. Marshakov, A. Mironov and D. Vasiliev, *Complex geometry of matrix models*, `hep-th/0506075`.

[37] C. Faber, *Algorithms for computing intersection numbers on moduli spaces of curves, with an application to the class of the locus of Jacobains*, `alg-geom/9706006`.

[38] Y. Makeenko, A. Marshakov, A. Mironov and A. Morozov, *Continuum versus discrete Virasoro in one matrix models*, *Nucl. Phys.* **B 356** (1991) 574.

[39] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311 [`hep-th/9309140`];
M. Aganagic, R. Dijkgraaf, A. Klemm, M. Mariño and C. Vafa, *Topological strings and integrable hierarchies*, *Commun. Math. Phys.* **261** (2006) 451 [`hep-th/0312085`];
M. Mariño, *Open string amplitudes and large order behavior in topological string theory*, *JHEP* **03** (2008) 060 [`hep-th/0612127`];
B. Eynard, M. Mariño and N. Orantin, *Holomorphic anomaly and matrix models*, *JHEP* **06** (2007) 058 [`hep-th/0702110`];
V. Bouchard, A. Klemm, M. Mariño and S. Pasquetti, *Remodeling the B-model*, `arXiv:0709.1453`.